Conceptualizing life annuities as fuzzy random variables – an exploratory study

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ABSTRACT

In the conclusion to Shapiro (2013), where future lifetime was modeled as a fuzzy random variable (FRV), it was suggested that a logical next step would be to merge a FRV future lifetime with a FRV interest rate, and to use them to evaluate various accumulation and discount models in insurance. Andrés-Sánchez and González-Vila Puchades (2017), who constructed a FRV by merging a fuzzy discount rate with a stochastic mortality rate, expressed a similar sentiment. This paper is a follow-up to those suggestions. Building on Shapiro (2013) and Wang (2019), we show how to conceptualize life annuities when both future lifetime and interest rates are fuzzy random variables.

1. Introduction

The purpose of this study is to conceptualize both the interest rate and the future lifetime as FRVs, and then to use them to model life annuities. The essential feature of this model is that it explicitly merges the stochastic component of mortality with its fuzzy component and the stochastic component of the interest rate with its fuzzy component. In this sense, both the mortality and interest components of the model are said to be granulated.

We focus on the variability inherent in the life annuities; only traditional life annuities are considered. L-R type fuzzy numbers are used for modeling, so we discuss what they are and how they are applied. This version of our model should be viewed as a prototype, and preliminary.

2. Some previous key life annuity models

Figure 1 shows the chronology of some previous key contributions to life annuity modeling.

Starting with Ulpian (211) [see Chapter 6 in Poitras et al. (2000)], who developed a crude life annuity model based on median life expectancy, and seems to have been among the earliest to quantify an annuity. Then de Witt (1671) advocated the use of age at issue for life annuities. Apparently, at the time, the present value of annuities were based on a hypothetical duration, which was an advantage for those who would game the system. Halley (1693), of comet fame, introduced mortality rates (via the Breslau mortality table) into the annuity computation.

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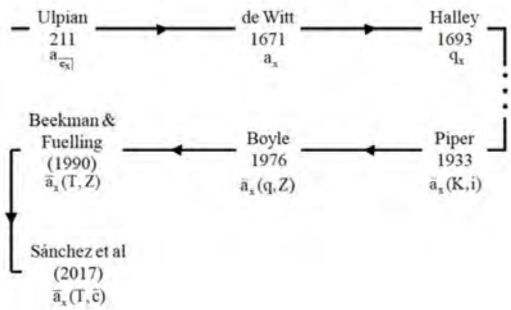

Figure 1: Chronology of some previous key life annuity models

It took almost 250 years before an analysis of the randomness associated with annuities began to appear in the literature. Piper (1933) was among the first in that endeavor when he examined the distribution for a life annuity based on a random future lifetime. Figure 2 shows how Piper conceptualized this distribution.

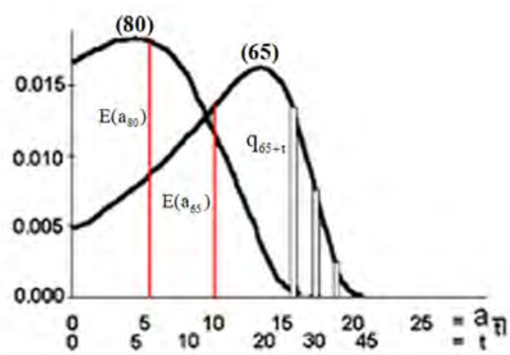

Figure 2: Distributions of a life annuity

The curve on the left and the first vertical line represent the distribution for a life annuity for a life aged 80 and its expected value, respectively. The area to the right of the vertical line represents the probability of the annuity exceeding its mean. Similarly, the second curve and the second vertical line represent the distribution for a life annuity for a life aged 65 and its expected value, respectively. The rectangles on the right of the figure represent probabilities of dying during a given year of age (${}_{t|}q_x$). These are interesting representations, since they were accomplished long before computers were available.

Piper (1933) focused on the mortality aspects of randomness. Boyle (1990), in contrast, took the mortality rate as given and investigated the impact of stochastic rates of return, and the associated mean and standard deviation of annuities certain and life annuities. In passing, Boyle (1976: 700) made the observation that "It would be possible to set up a model where the ${}_{t|}q_x$ themselves are regarded as random … ." So he recognized that a model involving both a random interest component and a random mortality component would be a future innovation.

This came to pass when Beekman and Fuelling (1990), among others, investigated the impact on annuities when both the interest rate and the future life time were random variables. Finally, with respect to the fuzzy random variable (FRV) context, we note that Sánchez et al (2017) investigated life annuities based on their version of FRVs, which combined a random mortality component with a fuzzy discount rate (ie., their FRV was not granulated).

3. The present value of a random future payment stream

In this study, we build on Beekman and Fuelling's (1990) model for the present value of a random future payment stream, where:

n is the term of the annuity

δ is the constant force of interest

V(s) is a stochastic process, at time s, that perturbs δ

W(s) = δ + V(s), $0 \leqslant s \leqslant n$, are the variable forces of interest, [4]

Z(t)=$\int_0^t V(s)\,ds, 0 \leqslant t \leqslant n$, is the cumulative stochastic process. [5]

Then the force of interest accumulation function (Beekman and Fuelling's term), is given by:

$$\int_0^t W(s)ds = \delta t + Z(t),\ t \geqslant 0. \tag{1}$$

and the stochastic process $e^{-(\delta t + Z(t))}, t \geqslant 0$, can be viewed as a random discount function. The present value of the random future payment stream, per unit of payment, is given by

$$\bar{a}_{\overline{n}|W} = \int_0^n e^{-\int_0^t W(s)\,ds} dt = \int_0^n e^{-[\delta t + Z(t)]} dt, \tag{2}$$

which is what one would expect.

4. Incorporating the random future lifetime component

Following Beekman and Fuelling (1990: 190), the fixed term of the annuity depicted in (2), is replaced with the random future lifetime of the annuitant, T(x).

Then, writing T for T(x), the random present value of a future payment stream, per unit of payment, is given by

$$\bar{a}_{\overline{T}|W} = \int_0^T e^{-\int_0^t W(s)\,ds} dt = \int_0^T e^{-[\delta t + Z(t)]} dt \tag{3}$$

an extension, of which, to FRVs, forms the basis of our model.

Beekman and Fuelling noted that their model is encapsulated in the following Law of Iterated Expectations statement:

$$E[\bar{a}_{\overline{T}|W}] = E_W\, E_T[\bar{a}_{\overline{T}|W} | W] \tag{4}$$

That is, given W, the variable force of interest, the expected value of the annuity, relative to T, the future lifetime, is formulated. Then, taking the EV of that quantity, with respect to the variable force of interest, gives the expected value of the annuity, the LHS.

[4] We use W(s), rather than Beekman and Fuelling's R(s), since we reserve the letter R for L-R type fuzzy numbers.
[5] We use Z(t), rather than Beekman and Fuelling's X(t), since we reserve the letter X for the random age at death.

5. Notation used in this article

This section defines some of the notation is used in this article:

FV: Fuzzy Variable

RV: Random Variable

FRV: Fuzzy Random Variable

$T(x)$: RV future lifetime of a life aged x (abbreviated T)

$\tilde{T}(x)$: FRV future lifetime of a life aged x (abbreviated $\tilde{T}$)

$Z(t), t \geqslant 0:$ RV cumulative stochastic process (abbreviated Z)

$\tilde{Z}(t), t \geqslant 0:$ FRV cumulative stochastic process (abbreviated $\tilde{Z}$)

$\bar{a}_{\overline{T}|}(T, Z)$: continuous life annuity given $T(x)$ and $Z(t)$

$\bar{a}_{\overline{T}|}(\tilde{T}, \tilde{Z})$: continuous life annuity given $\tilde{T}(x)$ and $\tilde{Z}(t)$

$\bar{a}_x(T, Z):$ EV of a continuous life annuity given $T(x)$ and $Z(t)$

$\bar{a}_x(\tilde{T}, \tilde{Z}):$ EV of a continuous life annuity given $\tilde{T}(x)$ and $\tilde{Z}(t)$

We have already mentioned the first 7 items. The last 4 items will be used in the pages to follow, and constitute new notation.

In particular, the $\bar{a}_{\overline{T}|}(T, Z)$ on the 4th line from the bottom, denotes a continuous life annuity, based on the RVs T(x) and Z(t). In a similar vein, the $\bar{a}_x(\tilde{T}, \tilde{Z})$, on the bottom line, denotes the expected value of a continuous life annuity, based on the FRVs $\tilde{T}(x)$ and $\tilde{Z}(t)$.

6. Continuous life annuity as a RV

Given a random future lifetime, T(x), and a random cumulative stochastic process, Z(t), (5) shows that its definitional term (its 2nd term) morphs into its operational term (its 3rd term).

$$\bar{a}_x(T, Z) = E\Big[\int_0^{T(x)} e^{-(\delta t + Z(t))} dt\Big] = \int_0^{\infty} E(e^{-(\delta t + Z(t))})\, {}_t p_x dt \tag{5}$$

The validation of (5) is as follows:

$$\begin{aligned}
\bar{a}_x(T, Z) &= E\Big[\int_0^{T(x)} e^{-(\delta t + Z(t))} dt\Big] \qquad (6)\\
&= \int_0^{\infty} E(e^{-(\delta t + Z(t))} I_{\{t<T(x)\}}) dt \\
&= \int_0^{\infty} \Big[E(e^{-(\delta t + Z(t))} \cdot 1) P(I_{\{t<T(x)\}} = 1) \\
&\quad + E(e^{-(\delta t + Z(t))} \cdot 0) P(I_{\{t<T(x)\}} = 0)\Big] dt \\
&= \int_0^{\infty} \big[E(e^{-(\delta t + Z(t))})\big] {}_t p_x dt,
\end{aligned}$$

where I denotes an indicator function of a set, here

$$I_{\{t<T(x)\}} = \begin{cases} 1, & t < T(x) \\ 0, & t \geqslant T(x). \end{cases} \tag{7}$$

As indicated, we start with the definitional term on the RHS of line 1 of (6), which gives the EV of a continuous life annuity, whose term is T(x). Then, using the indicator function, we transition from line 1 to line 2 of (6), which is a key step. Finally, we morph into the intuitive operational statement in the last line of (6).

7. Continuous life annuity as a FRV

Given the RV version of the Beekman and Fuelling (1990) model, our study extends their model to a FRV context. That is, starting with (5), which is based on the RVs future lifetime and cumulative stochastic process, our goal is to extend the model along the lines of (8).

$$\bar{a}_x(\tilde{T}, \tilde{Z}) = E\Big[\int_0^{\tilde{T}(x)} e^{-(\delta t+\tilde{Z}(t))} dt\Big], \tag{8}$$

which is based on the FRV versions of the future lifetime,$\tilde{T}(x)$, and cumulative stochastic process, $\tilde{Z}(t)$.

We note that Wang (2019) did preliminary modeling along these lines.

8. FRV future lifetime of a life aged x

So what do we mean by a FRV future lifetime, $\tilde{T}(x)$? Here is one representation of that.[6]

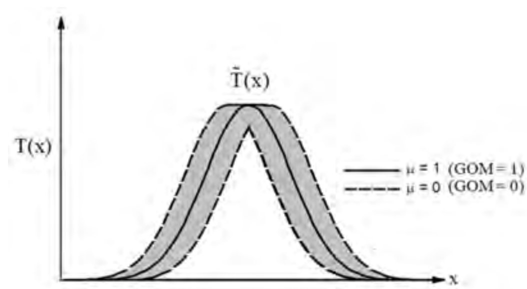

Figure 3: FRV future lifetime

The solid line represents the pdf of the RV future lifetime of a life aged x. Coinciding with that, is the mode of its fuzzy component, whose grade of membership, μ is equal to 1. The dashed lines on either side of the solid curve, represent a GOM of 0, $\mu = 0$, of the fuzzy component. The combination of the RV future lifetime and its FV component, together, form the FRV future lifetime.

A similar explanation holds for a FRV cumulative stochastic process,$\tilde{Z}$.

9. Dubois and Prade's L-R type fuzzy numbers

In this study, in order to improve the computational speed of our model, we use the Dubois and Prade (1980) L-R type fuzzy numbers. Sections 9 and 10 describe the nature of these L-R type fuzzy numbers, and how they are implemented.[7].

[6] Adapted from Möller et al. (2005)
[7] Sections 9 and 10 are based on Shapiro (2022)

Readers familiar with these topics can skip these sections.

Triangular fuzzy numbers (TFNs) are used to simplify the illustrations.

9.1. L-R type fuzzy numbers

Assuming a TFN, the general characteristics of the membership function (MF) of a L-R type fuzzy number are shown in Figure 4, where $\mu_{\tilde{u}}(x)$ denotes the membership function for a fuzzy variable u. [8]

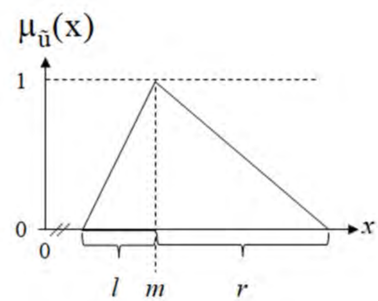

Figure 4: L-R type TFN

As indicated, the key components are:

m, the mode of the MF

l > 0, the left-hand spread of the MF

r > 0, the right-hand spread of the MF

We write ũ = (m, l, r)$_{LR}$ to denote the L-R type fuzzy number ũ.

Formally, a fuzzy number is said to be a L-R type FN, iff it satisfies (9).

$$\mu_{\tilde{u}}(x) = \begin{cases} L(\frac{m-x}{l}), & x \leqslant m, \quad l > 0, \\ R(\frac{x-m}{r}), & x \geqslant m, \quad r > 0. \end{cases} \tag{9}$$

Figures 5(a) and (b) are representations of the L and R-type shape functions referred to in (9).

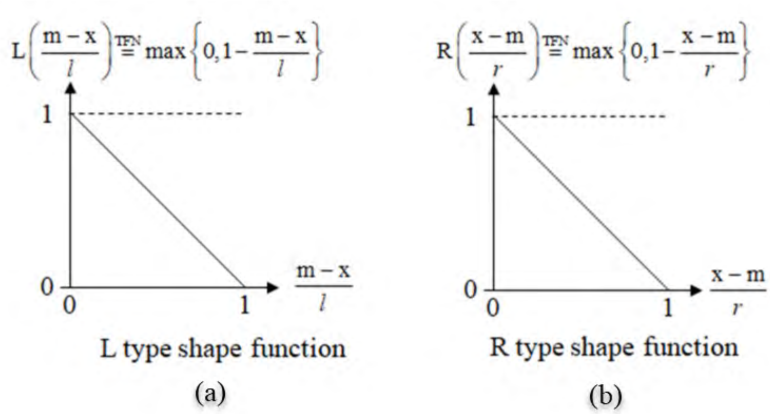

Figure 5: L-R type shape functions

Note that, since these are TFNs, they take the simple form of max[0, 1-y], where the value of y depends on whether it represents the left or right spread.

[8] A general limitation of L-R type fuzzy numbers is that they only apply to uni-modal fuzzy numbers. However, there is a work around for trapezoidal-type FNs, see Dubois and Prade (2000), p. 512.

As indicated, these shape functions:

Are functions on $[0, \infty)$

Are non-increasing

Are left continuous

Satisfy L(0) = R(0) = 1

Satisfy L(1) = R(1) = 0

Zimmermann (2001: 65) counsels that finding an appropriate function in a specific context may be a problem.

9.2. The inverted shape functions $L^{-1}(\alpha)$ and $R^{-1}(\alpha)$

Equation (10), which is based on the inf and sup of the α-cut of the fuzzy number ũ, and will be used in the pages to follow, depends on the inverted shape functions $L^{-1}(\alpha)$ and $R^{-1}(\alpha)$

$$\tilde{u}_\alpha = \left[m - l\, L^{-1}(\alpha),\ m + r\, R^{-1}(\alpha)\right], \alpha \in [0, 1]. \tag{10}$$

(11) defines the $L^{-1}(\alpha)$ function,

$$L^{-1}(\alpha) := \sup\{s \in \mathbb{R} | L(s) \geqslant \alpha\}, \tag{11}$$

while (12) defines the $R^{-1}(\alpha)$ function,

$$R^{-1}(\alpha) := \sup\{s \in \mathbb{R} | R(s) \geqslant \alpha\}. \tag{12}$$

Their representations are shown in Figures 6(a) and (b):

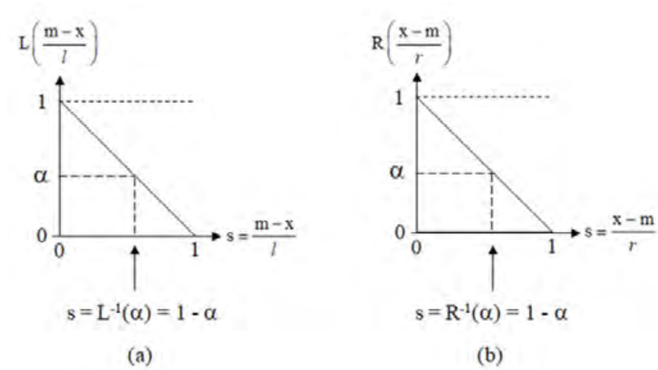

Figure 6: L-R type shape functions

As shown in the figure, since we are dealing with TFN, the inverted shape functions are simply equal to $1 - \alpha$.

Essentially, referring to (11) and Figure 6(a), $L^{-1}(\alpha)$ is the largest value on the horizontal axis for which L(s) $\geqslant \alpha$.

Similarly, referring to (12) and Figure 6(b), $R^{-1}(\alpha)$ is the largest value on the horizontal axis for which R(s) $\geqslant \alpha$.

9.3. Using inverted shape functions to create α-cuts

Given an L-R type FN and its parameters, $\tilde{u} = (m, l, r)_{LR}$, the related inverted functions, $L^{-1}(\alpha)$ and $R^{-1}(\alpha)$, can be used to recover the MF. Essentially, as shown in (13)

$$\tilde{u}_\alpha = \left[m - l\ L^{-1}(\alpha),\ m + r\ R^{-1}(\alpha)\right], \alpha \in [0, 1] \tag{13}$$

and Figure 7, the inf of the MF at α is given by m - l $L^{-1}(\alpha)$ and the sup of the MF at α is given by m + r $R^{-1}(\alpha)$.

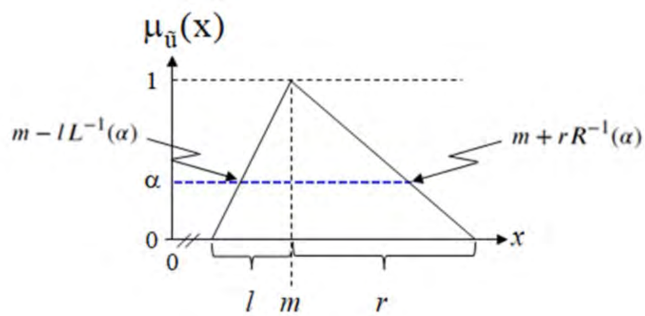

Figure 7: Creating α-cuts

The entire MF is constructed by juxtaposing each of these α slices, $0 < \alpha \leqslant 1$.

10. L-R type fuzzy number arithmetic

Given the general nature of L-R type fuzzy numbers, this section provides a brief overview of the arithmetic of L-R type fuzzy numbers that will be used in this article. The topics covered include: the addition of L-R type fuzzy numbers, the negative of an L-R type fuzzy number, the subtraction of L-R type fuzzy numbers, and the multiplication of L-R type fuzzy numbers.

10.1. The addition of L-R type fuzzy numbers

We turn first to the addition of two L-R type fuzzy numbers. (14) gives the formula for this addition:

$$(m_1, l_1, r_1)_{LR} \oplus (m_2, l_2, r_2)_{LR} = (m_1 + m_2, l_1 + l_2, r_1 + r_2)_{LR}, \tag{14}$$

The result is what you would expect, i.e., the modes are added together, as are the left and right spreads.

The more complicated equation has to do with the α-cuts, shown in (15):

$$\tilde{u}_\alpha = \left[(m_1 + m_2) - [l_1 L_1^{-1}(\alpha) + l_2 L_2^{-1}(\alpha)], (m_1 + m_2) + [r_1 R_1^{-1}(\alpha) + r_2 R_2^{-1}(\alpha)]\right] \tag{15}$$

If, as here, each FN has a unique MF, their inverted shape functions will also be different, and the new left and right spreads will be the sum of two unique components, as shown in (15).

Of course, if the FNs have the same MF or the MFs are TFNs, (15) could be simplified accordingly.

Figure 8 provides a simple example of fuzzy number addition, showing that the modes are added together (20+60=80), as are the left and right spreads, (10+20=30).

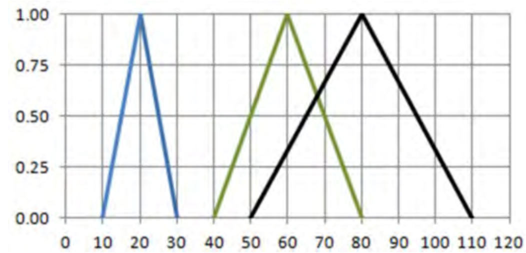

Figure 8: Example of fuzzy number addition

10.2. The negative of a L-R type FN

We turn now to the negative of a L-R type FN, which we use in our analysis. Its formula is given by (16):

$$-(m, l, r)_{LR} = (-m, r, l)_{RL} \tag{16}$$

As indicated, the resulting L-R type FN

Has a negative mode, -m, and

The references l and r exchange positions

The rationale for this result is conceptualized in Figure 9,

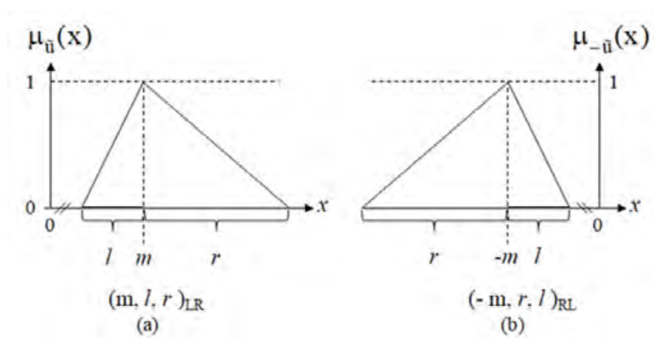

Figure 9: The negative of a L-R type FN

where Figure 9(a) depicts the original MF while Figure 9(b) depicts its negative, as a mirror image about the vertical axis.

10.3. The subtraction of an L-R type fuzzy number

(17) shows the formulas for the subtraction of two L-R type fuzzy numbers:

$$\begin{aligned} &(m_1, l_1, r_1)_{LR} \ominus (m_2, l_2, r_2)_{LR} \\ &= (m_1, l_1, r_1)_{LR} \oplus (-m_2, r_2, l_2)_{LR} \\ &= (m_1 - m_2, l_1 + r_2, r_1 + l_2)_{LR} \end{aligned} \tag{17}$$

The result is what one might expect, given the equations for the addition and the negative of a L-R type fuzzy number.

The α-cut of the subtraction of two L-R type fuzzy numbers is given by (18):

$$\tilde{u}_\alpha = \left[(m_1 - m_2) - [l_1 L_1^{-1}(\alpha) + r_2 R_2^{-1}(\alpha)], (m_1 - m_2) + [r_1 R_1^{-1}(\alpha) + l_2 L_2^{-1}(\alpha)]\right] \tag{18}$$

A simple example of the subtraction of two L-R type fuzzy numbers is given in Figure 10:

As mentioned previously, if the FNs have the same MF or the MFs are TFNs, (18) would be simplified accordingly.

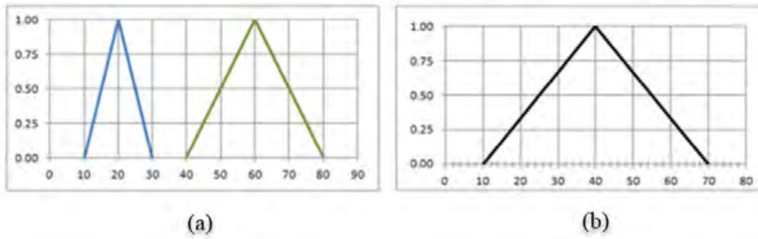

Figure 10: Subtraction of two L-R type fuzzy numbers

10.4. The multiplication of L-R type fuzzy numbers

This subsection shows the formulas associated with the multiplication of L-R type fuzzy numbers.

To begin, consider (19). Briefly, the fuzzy product shown in the LHS of (19) results in the 3 groupings on the RHS: the product of the modes, the terms involving the LH spreads, and the terms involving the RH spreads.

$$(m_1, l_1, r_1)_{LR} \otimes (m_2, l_2, r_2)_{LR} = (m_1\, m_2, m_1\, l_2 + m_2\, l_1\, + l_1\, l_2, m_1\, r_2 + m_2\, r_1 + r_1\, r_2)_{RL},\; m_1, m_2 > 0 \tag{19}$$

The α-cut is shown in (20):

$$\tilde{u}_\alpha = \begin{cases} (m_1 \times m_2) - [m_2\, l_1\, L_1^{-1}(\alpha) + m_1\, l_2\, L_2^{-1}(\alpha) + l_1\, l_2(L_1^{-1}(\alpha) \times L_2^{-1}(\alpha))], \\ (m_1 \times m_2) + [m_2\, r_1\, R_1^{-1}(\alpha) + m_1\, r_2\, R_2^{-1}(\alpha) + r_1\, r_2(R_1^{-1}(\alpha) \times R_2^{-1}(\alpha))] \end{cases} \tag{20}$$

Note, in particular, that the last term of each line of (20) involves the products of the inverted shape functions, along with the product of their spreads.

Finally, Figure 11 provides an example of the multiplication of L-R type fuzzy numbers.

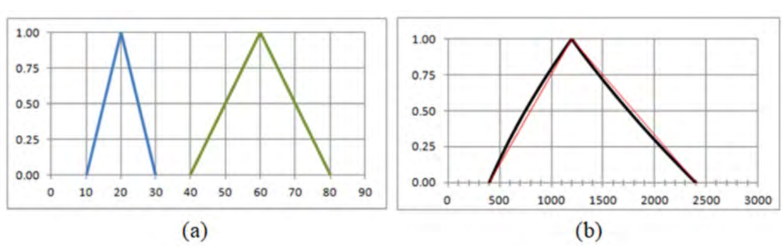

Figure 11: Example of Multiplication of L-R FNs

Notice the bowled nature of the product (the dark line) in Figure 11(b).

11. Assumptions used in this study

We assume that:

The FRVs future lifetime, $\tilde{T}$(x) and cumulative interest rate, $\tilde{Z}$(t) can be represented as L-R type FRNs. Specifically, taking the RVs T(x) and Z(t) as modes,

$$\tilde{T}(x) = (T(x), l_T, r_T) \tag{21}$$

$$\tilde{Z}(t) = (Z(t), l_Z, r_Z) \tag{22}$$

V(s), the stochastic process, at time s, that perturbs δ, the constant force of interest, is an iid random variable.

T(x), $0 \leqslant x < \omega$, is independent of Z(t), $0 \leqslant t < \infty$,

$\tilde{T}(x), 0 \leqslant x < \omega$, is independent of $\tilde{Z}(t)$, $0 \leqslant t < \infty$.

12. Continuous life annuity as a FRV (cont)

With sections 9 and 10 as background, we return to the problem at hand. Given the Beekman and Fuelling version of the continuous life annuity as a RV, (5), our goal is to extend their model to the FRV context, given by (8).

13. The α-cut for a FRV version of a continuous life annuity

The general form of the α-cut for a FRV version of a continuous life annuity, for a life aged x, is given by (23):

$$\bar{a}_x(\tilde{T},\tilde{Z})_\alpha = \Big[\inf \bar{a}_x(\tilde{T},\tilde{Z})_\alpha, \sup \bar{a}_x(\tilde{T},\tilde{Z})_\alpha\Big]. \tag{23}$$

Consider first the inf term of (23), which is expressed in (24). The basic components are: the α-cut of the FRV future lifetime for a life aged x, $\tilde{T}_\alpha$(x), and the α-cut of the FRV cumulative stochastic process at time t, $\tilde{Z}_\alpha$(t).

$$\begin{aligned}\inf \bar{a}_x(\tilde{T},\tilde{Z})_\alpha = \min\Big\{ & E\Big(\int_0^{\inf \tilde{T}_\alpha(x)} e^{-\inf(\delta t+\tilde{Z}_\alpha(t))}dt\Big),\\ & E\Big(\int_0^{\inf \tilde{T}_\alpha(x)} e^{-\sup(\delta t+\tilde{Z}_\alpha(t))}dt\Big),\\ & E\Big(\int_0^{\sup \tilde{T}_\alpha(x)} e^{-\inf(\delta t+\tilde{Z}_\alpha(t))}dt\Big),\\ & E\Big(\int_0^{\sup \tilde{T}_\alpha(x)} e^{-\sup(\delta t+\tilde{Z}_\alpha(t))}dt\Big)\Big\}.\end{aligned} \tag{24}$$

By implementing Zadeh's extension principle, we choose the minimum of the 4 EV's in (24) as the overall inf.

In contrast, to obtain the sup, we choose the maximum of the 4 EV's in (25):

$$\begin{aligned}\sup \bar{a}_x(\tilde{T},\tilde{Z})_\alpha = \max\Big\{ & E\Big(\int_0^{\inf \tilde{T}_\alpha(x)} e^{-\inf(\delta t+\tilde{Z}_\alpha(t))}dt\Big),\\ & E\Big(\int_0^{\inf \tilde{T}_\alpha(x)} e^{-\sup(\delta t+\tilde{Z}_\alpha(t))}dt\Big),\\ & E\Big(\int_0^{\sup \tilde{T}_\alpha(x)} e^{-\inf(\delta t+\tilde{Z}_\alpha(t))}dt\Big),\\ & E\Big(\int_0^{\sup \tilde{T}_\alpha(x)} e^{-\sup(\delta t+\tilde{Z}_\alpha(t))}dt\Big)\Big\}.\end{aligned} \tag{25}$$

In the pages to follow, we will use L-R type FNs to evaluate the inf and sup of $\bar{a}_x(\tilde{T},\tilde{Z})_\alpha$.

14. The morphing of the $\tilde{T}_\alpha(x)$ portion of the integrals in §13

The following summarizes the steps in the morphing of the integral of 0 to the inf of the α-cut of the FRV future lifetime, line 1 of (26), to its implementation form, line 4 of (26).[9]

[9] Only the portion of the integral involving $\tilde{T}_\alpha(x)$ is included in this equation and the "$\cdots$" signifies that the discount portion of the equation is suppressed (not shown).

$$
\begin{aligned}
& E \int_0^{\inf \tilde{T}_\alpha(x)} \cdots \, dt, \qquad (26)\\
&= E \int_0^{T(x) - l_T \, L^{-1}(\alpha)} \cdots \, dt,\\
&= \int_0^{\infty} P(t < T(x) - l_T \, L^{-1}(\alpha)) \cdots \, dt,\\
&= \int_0^{\infty} {}_{t + l_T \, L^{-1}(\alpha)} p_x \cdots \, dt
\end{aligned}
$$

Note that:

Line 2 of (26) replaces the inf of the α-cut of the FRV with its L type FN representation,

Line 3 of (26) substitutes an equivalent probability statement for line 2, and

Finally, line 4 of (26) puts it into a probability of persisting configuration.

Similarly, the integral of 0 to the sup of the α-cut of the FRV future lifetime, line 1 of (27), can be morphed into its implementation form, line 4 of (27).

$$
\begin{aligned}
& E \int_0^{\sup \tilde{T}_\alpha(x)} \cdots \, dt, \qquad (27)\\
&= E \int_0^{T(x) + r_T \, R^{-1}(\alpha)} \cdots \, dt,\\
&= \int_0^{\infty} P(t < T(x) + r_T \, R^{-1}(\alpha)) \cdots \, dt,\\
&= \int_0^{\infty} {}_{t - r_T \, R^{-1}(\alpha)} p_x \cdots \, dt
\end{aligned}
$$

15. Discounted value in L-R form

Similarly, the inf of the discounted value can be morphed into its L-R form:

$$
\begin{aligned}
& e^{- inf \, (\delta \, t + \tilde{Z}_\alpha(t))} \qquad (28)\\
&= e^{-\delta \, t} e^{- inf \, \tilde{Z}_\alpha(t)}\\
&= e^{-\delta \, t} e^{-[Z(t) - l_Z(t) \, L^{-1}(\alpha)]}\\
&= e^{-\delta \, t} e^{[-Z(t) - r_Z(t) \, R^{-1}(\alpha)]}
\end{aligned}
$$

Finally we morph the sup of the discounted value into its L-R form:

$$
\begin{aligned}
& e^{- sup \, (\delta \, t + \tilde{Z}_\alpha(t))} \qquad (29)\\
&= e^{-\delta \, t} e^{- sup \, \tilde{Z}_\alpha(t)}\\
&= e^{-\delta \, t} e^{-[Z(t) + r_Z(t) \, R^{-1}(\alpha)]}\\
&= e^{-\delta \, t} e^{[-Z(t) + l_Z(t) \, L^{-1}(\alpha)]}
\end{aligned}
$$

16. α-cuts for a FRV continuous life annuity in terms of L-R type FNs

This section formulates the α-cuts for a FRV continuous life annuity in terms of L-R type FNs.

Once again, we start with the RHS of (23) as an explicit statement of the boundary conditions, that is, of the inf and sup of the α-cuts.

Then (30) seeks the minimum of the 4 integrals of the products of persisting and expected discount:

$$\inf \bar{a}_x(\tilde{T},\tilde{Z})_\alpha = \min\Big\{ \int_0^\infty {}_{t+l_T(x)\,L^{-1}(\alpha)}\,p_x \;\; e^{-\delta t}\; E e^{[-Z(t)-r_Z(t)\,R^{-1}(\alpha)]}\,dt, \tag{30}$$
$$\int_0^\infty {}_{t+l_T(x)\,L^{-1}(\alpha)}\,p_x \;\; e^{-\delta t}\; E e^{[-Z(t)+l_Z(t)\,L^{-1}(\alpha)]}\,dt,$$
$$\int_0^\infty {}_{t-r_T(x)\,R^{-1}(\alpha)}\,p_x \;\; e^{-\delta t}\; E e^{[-Z(t)-r_Z(t)\,R^{-1}(\alpha)]}\,dt,$$
$$\int_0^\infty {}_{t-r_T(x)\,R^{-1}(\alpha)}\,p_x \;\; e^{-\delta t}\; E e^{[-Z(t)+l_Z(t)\,L^{-1}(\alpha)]}\,dt\Big\}$$

Similarly, (31) seeks the max of the 4 integrals of the products of persisting and expected discount

$$\sup \bar{a}_x(\tilde{T},\tilde{Z})_\alpha = \max\Big\{ \int_0^\infty {}_{t+l_T(x)\,L^{-1}(\alpha)}\,p_x \;\; e^{-\delta t}\; E e^{[-Z(t)-r_Z(t)\,R^{-1}(\alpha)]}\,dt, \tag{31}$$
$$\int_0^\infty {}_{t+l_T(x)\,L^{-1}(\alpha)}\,p_x \;\; e^{-\delta t}\; E e^{[-Z(t)+l_Z(t)\,L^{-1}(\alpha)]}\,dt,$$
$$\int_0^\infty {}_{t-r_T(x)\,R^{-1}(\alpha)}\,p_x \;\; e^{-\delta t}\; E e^{[-Z(t)-r_Z(t)\,R^{-1}(\alpha)]}\,dt,$$
$$\int_0^\infty {}_{t-r_T(x)\,R^{-1}(\alpha)}\,p_x \;\; e^{-\delta t}\; E e^{[-Z(t)+l_Z(t)\,L^{-1}(\alpha)]}\,dt\Big\}$$

Essentially, the solutions to (30) and (31) give us our bottom line.

17. Comments

The purpose of this article was to discuss the conceptualizing of life annuities as fuzzy random variables.

The essential features of our model was that:

It explicitly merged the stochastic components of mortality and interest with their fuzzy component, and

L-R type fuzzy numbers were employed.

In this article, we focused on methodology, and an explanation of that methodology.

Subsequent studies will address the implementation of the methodology.

References

Andrés-Sánchez, J., González-Vila Puchades, L., 2017. The valuation of life contingencies: A symmetrical triangular fuzzy approximation. Insurance: Mathematics and Economics 72, 83–94.

Beekman, J.A., Fuelling, C.P., 1990. Interest and mortality randomness in some annuities. Insurance: Mathematics and Economics 9, 185–196.

Boyle, P.P., 1990. Rates of return as random variables. Journal of Risk and Insurance 43, 693–713.

Dubois, D., Prade, H., 1980. Fuzzy Sets and Systems: Theory and Applications. Academic Press, New York.

Dubois, D., Prade, H., 2000. Fundamentals of fuzzy sets. volume 7. Springer Science & Business Media.

Halley, E., 1693. An estimate of the degrees of the mortality of mankind, drawn from the curious tables of the births and funerals at the city of breslaw. Philosophical Transactions 17, 596–610.

Möller, B., Graf, W., Hoffmann, A.and Steinigen, F., 2005. Numerical simulation of rc structures with textile reinforcement. Computers and Structures 83, 1659–1688.

Piper, K.B., 1933. Contingency reserves for life annuities. Transactions of the Actuarial Society of America 34, 240–49.

Poitras, G., et al., 2000. The early history of financial economics, 1478–1776. Edward Elgar Publishing.

Shapiro, A.F., 2013. Modeling future lifetime as a fuzzy random variable. Insurance: Mathematics and Economics 53, 864–870.
Shapiro, A.F., 2022. Conceptualizing Dubois and Prade's L-R type fuzzy numbers and their implementation. Penn State Risk Management Research Center.
Wang, D., 2019. A net premium model for life insurance under a sort of generalized uncertain interest rates, in: Destercke, S., Denoeux, T., Gil, M.Á., Grzegorzewski, P., Hryniewicz, O. (Eds.), Uncertainty Modelling in Data Science, Springer International Publishing, Cham. pp. 224–232.
de Witt, J., 1671. Treatise on life annuities. Reprinted in The Journal of the Institute of Actuaries 44, 500.

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