An Integro-differential Equation for a Sparre Andersen Model with Investments

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- Classical actuarial problem - the collective risk Sparre Andersen model .
- Additional non-traditional feature, investments in a risky asset with returns modeled by a stochastic process.
- Focus of the analysis: the probability of ruin.
- Decay of the probability of ruin in the case of Erlang(n) distribution for inter-claims returns modeled by a geometric Brownian motion.

Sparre Andersen model

$$U_t = u + ct - \sum_{k=1}^{N(t)} X_k$$

- u -initial surplus
- c -premium rate
- X_k iid - "light" claims - $F_X \sim$ exponentially bounded tail
- $N(t)$ - renewal process
- $T_1, T_2, \cdots$ times when claims occur
- $\tau_0 = 0, \tau_n = T_n - T_{n-1}$ inter-arrival times, independent, identical distributed r.v.

Sparre Andersen model with investments

- Consider that the company invests all its money, continuously, in a risky asset modeled by a non-negative stochastic process.
- NOTE: The ruin may happen only at the time of a claim, T_k.
- The model

$$\boxed{U_k = Z_{\tau_k}^{U_{k-1}} - X_k}$$

is a discrete Markov process, where $U_k = U_{T_k}$.

Definitions

The time of ruin:

$$T_u = \inf_{t \geq 0} \{U(t) < 0 \mid U(0) = u\}$$

The probability of ruin with infinite horizon:

$$\Psi(u) = P(T_u < \infty).$$

Objectives

1. An equation for the probability of ruin in the Sparre Andersen model with investments
2. Particular case: Investigate the decay of the probability of ruin if the interarrival times are Erlang (n, β), with returns from investments modeled by GBM (a, σ^2).

Main tools

1. Integro-differential equation: generators arguments
2. Decay: Karamata-Tauberian arguments

Assumptions

- $(X_k)_k$- claim sizes - "light" or well-behaved distributions F_X with exponentially bounded tail

$$1 - F_X(x) \leq ce^{-ax}$$

for some a and c and for all $x \geq 0$.

- $(\tau_k)_k$- inter-arrival times - f_τ satisfies an ODE with constant coefficients

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = 0$$

Example: $f_\tau(t) = \beta e^{-\beta t}$ then $\mathcal{L}(\frac{d}{dt})f_\tau(t) = (\frac{d}{dt} + \beta)f_\tau(t) = 0$

- Z_t^u - returns from investments up to time t, starting with an initial capital u- the company invests all its money, continuously into a risky asset modeled by a non-negative stochastic process with infinitesimal generator A

Transition operator of the discrete Markov process For our discrete Markov process $U_0, U_1, U_2, \cdots$ (where $U_k = U_{T_k}$), on the set of all real-valued, bounded, Borel measurable functions g, define the transition operator

$$T_k g(u) = E(g(U_k) \mid U_0 = u) = E_u g(U_k).$$

Then $M_n = f(U_n) - \sum_{k=0}^{n-1}(T_1 - I)f(U_k)$ is a martingale.

Proof: $E(M_{n+1} \mid \sigma(U_0, U_1, \cdots U_n)) = E(g(U_{n+1}) \mid U_0, U_1, \cdots U_n) - \sum_{k=0}^{n}(T_1 - I)g(U_k) = T_1 g(U_n) - T_1 g(U_n) + g(U_n) - \sum_{k=0}^{n-1}(T_1 - I)g(U_k) = M_n$.

Theorem

If f_τ satisfies the ODE with constant coefficients

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = 0$$

and

1. $f_\tau^{(k)}(0) = 0$, the $k-$th derivatives of f_τ, for $k = 0, \cdots, n-2$
2. $\lim_{x\to\infty} f^{(k)}(x) = 0$, for $k = 0, \cdots, n-1$

then for any $g \in \mathcal{D}_{A^{(n)}}$

$$\boxed{\mathcal{L}^*(A)T_1 g(u) = f_\tau^{(n-1)}(0)\int_0^\infty g(u-x)f_X(x)dx}$$

where A denotes the infinitesimal generator of the investment process Z_t, n represents the order of the ODE with constant coefficients satisfied by f_τ.

Relation to the ruin probability

Theorem. Assume that on the event $\{T_u = \infty\}$, $U_t \to \infty$ as $t \to \infty$. If $g \in \mathcal{D}_{A_U}$ satisfies

$$\boxed{\mathcal{L}^*(A)g(u) = f_{\tau}^{(n-1)}(0) \int_0^{\infty} g(u-x) f_X(x) dx}$$

together with the boundary conditions

$$g(u) = 1 \quad \text{if} \quad u < 0$$

$$\lim_{u \to \infty} g(u) = 0$$

then

$$g(u) = P(T_u < \infty)$$

Sketch of proof:

- $g(U_k)$ is a martingale, T_u stopping time
- $g(u) = E_u g(U_{T_u \wedge T_k}) =$
  $E_u g(U_{T_u \wedge T_k} 1_{\{T_u < T_k\}}) + E_u g(U_{T_u \wedge T_k} 1_{\{T_u > T_k\}}) =$
  $g(U_{T_u}) P(T_u < T_k) + g(U_{T_k}) P(T_u > T_k)$ (let $t \to \infty$)
- $g(u) = 1 * P(T_u < \infty) + 0 * P(T_u > \infty) = P(T_u < \infty)$

Examples

Integro-differential equation for Cramer Lundberg model - exp(β)

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = (\frac{d}{dt} + \beta)f_\tau(t) = 0 \implies \mathcal{L}^*(\frac{d}{dt}) = (-\frac{d}{dt} + \beta)$$

therefore

$$\mathcal{L}^*(A)\Psi(u) = (-A + \beta)\Psi(u) = \beta \int_0^\infty \Psi(u-x)f_X(x)dx$$

Examples

If no investments, $A = c\frac{d}{du}$,

$$(-c\frac{d}{du} + \beta)\Psi(u) = \beta \int_0^{\infty} \Psi(u-x) f_X(x) dx$$

$$\Psi'(u) = \frac{\beta}{c}\Psi(u) - \frac{\beta}{c} \int_0^{\infty} \Psi(u-x) f_X(x) dx$$

Particular case

- $f_X \sim$ finite moments in the neighborhood of the origin
- $f_\tau \sim$ Erlang (n, β) - $\mathcal{L}(\frac{d}{dt})f_\tau(t) = (\frac{d}{dt} + \beta)^n f_\tau(t) = 0$
- $Z \sim \text{GBM}(a, \sigma^2)$ returns,

$$\boxed{dZ = (c + aZ)dt + \sigma Z dW_t}$$

$$A = (c + au)\frac{d}{du} + \frac{\sigma^2}{2}u^2\frac{d^2}{du^2}$$

Then the surplus model is:

$$U(t) = u + ct + a\int_0^t U(s)ds + \sigma\int_0^t U(s)dW_S - \sum_0^{N(t)} X_k.$$

Integro-differential equation for Erlang(n) with investments

The integro-differential equation for a Sparre Andersen model when the time in between claims is Erlang(n, β)

$$(-A+\beta)^n \Psi(u) = \beta^n \int_0^\infty \Psi(u-x) f_X(x)dx$$

together with the boundary conditions for Ψ.
If the investments are made in a stock modeled by a geometric brownian motion

$$(-(c+au)\frac{d}{du} - \frac{\sigma^2 u^2}{2}\frac{d^2}{du^2} + \beta)^n \Psi(u) = \beta^n \int_0^\infty \Psi(u-x) f_X(x)dx$$

Then the decay of the probability of ruin is algebraic

$$\lim_{u\to\infty} \Psi(u) u^{-1+\frac{2a}{\sigma^2}} = K_n$$

for (small volatility) $1 < \frac{2a}{\sigma^2} < 2$.

Steps in establishing the algebraic decay rate

1. Take Laplace transform
2. Regularity at zero of the homogeneous ODE obtained in the Laplace side implies that $\hat{\Psi}(s) = s^\rho \sum_{k=0}^{\infty} c_k s^k$.
3. Karamata -Tauberian arguments

Laplace transform

- *Erlang*(n, β)

$$(-\hat{A}+\beta)^n \hat{\Psi}(s) = \beta^n \hat{f}_X \hat{\Psi}(s)$$

$$(-1)^n \hat{A}^n \hat{\Psi}(s) + \cdots + \beta^n = \beta^n \hat{\Psi}(s) \hat{f}_X(s) + \beta^n (\frac{1}{s} - \frac{\hat{f}_X(s)}{s})$$

- $2n$-th order ODE:

$$y^{(2n)} + p_1(s) y^{(2n-1)} + p_2(s) y^{(2n-2)} + \cdots + p_{2n}(s) y = p_{2n+1}(s)$$

- regularity at zero

$$\implies \hat{\Psi}(s) = s^\rho \sum_{k=0}^{\infty} c_k s^k$$

Regularity at zero

Determine ρ :

- The coefficient of the s^ρ term should be zero, i.e.

$$(-\delta + \beta)^n - \beta^n = 0$$

  where

$$\delta = [\sigma^2(\rho + 2) - a](\rho + 1)$$

- For $k = 0, 1, 2, \cdots, n-1$

$$\delta = \beta(1 - e^{\frac{2\pi ik}{n}})$$

- distinguish two cases, n odd or even

Case 1. $n =$ odd

$$\rho_1 = 0$$
$$\rho_2 = -2 + \frac{2a}{\sigma^2}$$

$$\rho_1 \leq \rho_2$$

- ρ_1 doesn't produce decay of the probability of ruin
- ρ_2 is the leading term,

$$\lim_{s \to 0} \hat{\Psi}(s) s^{2 - \frac{2a}{\sigma^2}} = K_n$$

$$\Longrightarrow \lim_{u \to \infty} \Psi(u) u^{-1 + \frac{2a}{\sigma^2}} = K_n$$

for $1 < \frac{2a}{\sigma^2} < 2$.

Case 2. n =even

$$\rho_1 = 0$$

$$\rho_2 = -2 + \frac{2a}{\sigma^2}$$

$$\rho_{3,4} = \frac{\rho_2 - 1}{2} \pm \sqrt{(\frac{\rho_2 + 1}{2})^2 + \frac{4\beta}{\sigma^2}}$$

$$\rho_4 \leq \rho_1 \leq \rho_2 \leq \rho_3$$

- ρ_4, ρ_1 do note produce decay of the probability of ruin
- By Karamata arguments and ordering of the ruin probabilities for Erlang of different n it can be shown that for any n,

$$\lim_{u \to \infty} \Psi(u) u^{-1+\frac{2a}{\sigma^2}} = K_n$$

for $1 < \frac{2a}{\sigma^2} < 2$.

Conclusions

1. For a Sparre Andersen model, perturbed by a stochastic process, a very general integro-differential equation for the ruin probability can be written, if the inter-claim arrivals are mixture of Erlangs.
2. For any n, the Sparre Andersen model with inter-arrival times distributed Erlang (n) and investments in a stock modeled by a GBM with small volatility, has an algebraic decay rate, depending on the parameter of the investments only.
3. Conjecture: in the case of high volatility, $\sigma^2 > 2a$, the ruin is certain.

Future questions

1. f_τ satisfies an ODE with polynomial coefficients
2. $f_\tau \sim Gamma(\alpha, \beta)$
3. Gerber-Shiu functions
4. Optimal investment strategy

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