Ruin with Delayed Claims and Investments Actuarial Research Conference - Drake University - August 1 2023

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$$U_t = u + ct + a\int_0^t U_s \mathrm{d}s + \sigma \int_0^t U_s \mathrm{d}W_s - \sum_{k=1}^{N(t)} X_k$$

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Example (Dassios-Zhao (2013)). Case of no-investments ($a = 0 = \sigma$). Asymptotic behavior of Ruin Probability as $u \to \infty$. Decrease in probability of ultimate ruin is independent of initial capital:

$$\frac{\psi(u, t)}{\psi(u, \infty)} \asymp e^{-cR\int_t^\infty (1 - L(s))\mathrm{d}s},$$

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$$\psi(u, \infty) \asymp \frac{c - \lambda\mu}{\lambda \int_0^\infty x e^{Rx} \mathrm{d}F(x) - c} e^{-Ru}$$

Equation for $\psi(u, t)$

Theorem (L-T). Assuming regularity, ψ satisfies the IDPE

$$\frac{\partial \psi}{\partial t} + (c + au)\frac{\partial \psi}{\partial u} + \frac{u^2}{2\sigma^2}\frac{\partial^2 \psi}{\partial u^2} + \lambda L(t) G(t, u, F, \psi) = 0. \qquad (1)$$

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Idea of proof Fix (u,t) and let $h>0$,

$$\begin{aligned}\psi(u,t) &= \mathbb{E}(1_{[\tau_t<\infty]}1_{[N(t+h)-N(t)=0]}|U_t=u) + \mathbb{E}(1_{[\tau_t<\infty]}1_{[N(t+h)-N(t)=1]}|U_t=u)\\ &\quad+\mathbb{E}(1_{[\tau_t<\infty]}1_{[N(t+h)-N(t)>1]}|U_t=u) \equiv I + II + III\\ I &= \mathbb{E}(1_{[\tau_{t+h}<\infty]}|U_t=u, 1_{[N(t+h)-N(t)=0]})\mathbb{P}(N(t+h)-N(t)=0)\end{aligned}$$

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&= \mathbb{E}(1_{[\tau_{t+h}<\infty]}|U_{t+h}=Z_h^u)\mathbb{P}(N(t+h)-N(t)=0) \\
&\asymp \mathbb{E}(\psi(Z_h^u, t+h))\,(1-\lambda L(t)h).
\end{aligned}$$

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Thus, using Itö,

$$\begin{aligned} \lim_{h\to 0}\frac{I-\psi(u,t)}{h} &= \lim_{h\to 0}\frac{\mathbb{E}(\psi(Z_h^u,t+h))-\psi(u,t)}{h} - \lambda L(t)\lim_{h\to 0}\mathbb{E}(\psi(Z_h^u,t+h)) \\ &= \frac{\partial \psi}{\partial t} + (c+au)\frac{\partial \psi}{\partial u} + \frac{\sigma^2 u^2}{2}\frac{\partial^2 \psi}{\partial u^2} - \lambda L(t)\psi(u,t) \end{aligned}$$

Equation for $\psi(u, t)$ - Continued

$$II = \int_t^{t+h} \int_0^{\infty} \mathbb{E}(1_{[\tau_s < \infty]} | (\tilde{T}_1) \in \mathrm{d}s, X_1 \in \mathrm{d}x, U_t = u) \mathrm{d}F_X(x)$$

$$= \int_t^{t+h} \mathbb{E} \left(\int_0^{Z^u_{s-t}} \psi(Z^u_{s-t} - x, s) \mathrm{d}F_X(x) + (1 - F_X(Z^u_{s-t})) \right) \mathrm{d}F_{\tilde{T}}(s)$$

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\end{aligned}
$$

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$$
\lim_{h \to 0} \frac{II}{h} = \left(\int_0^u \psi(u - x, t)\mathrm{d}F_X(x) + (1 - F_X(u)) \right) \lambda L(t)
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Since $III = o(h)$, the IPDE satisfied by ψ follows.

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Finally, $\lim_{t\to\infty} L(t) = 1, \lim_{t\to\infty} \psi_t = 0$, $\psi(t, \infty)$ satisfies with
$G_\infty(u, F, \psi) = \int_0^u \psi(u - x, \infty)\mathrm{d}F(x) + (1 - F(u)) - \psi(u, \infty)$,

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Finally, $\lim_{t\to\infty} L(t) = 1, \lim_{t\to\infty}\psi_t = 0$, $\psi(t,\infty)$ satisfies with $G_\infty(u,F,\psi) = \int_0^u \psi(u-x,\infty)\mathrm{d}F(x)+(1-F(u))-\psi(u,\infty)$,

$$(c+au)\frac{\partial\psi}{\partial u}+\frac{u^2}{2\sigma^2}\frac{\partial^2\psi}{\partial u^2}+\lambda G_\infty(u,F,\psi)=0 \qquad (2)$$

as claimed

Bounded delay, $L(t) = 1 \forall t > t^*$

For $t > t^*$ the solution of (1) is given by the time independent solution of (2).

Mild Formulation of IPDE (1) Using Feynman-Kac formula, regarding the term $\lambda L(t) G(t, u, F, \psi)$ as a 'forcing' term, ψ satisfies

$$\psi(u, t) = \mathbb{E}_u \left(\lambda \int_t^{t^*} L(r) G(Z_r, r, F, \psi) \mathrm{d}r + \psi(Z_{t^*}, \infty) \right) \qquad (3)$$

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$$\psi(u, t) = \mathbb{E}_u \left(\lambda \int_t^{t^*} L(r) G(Z_r, r, F, \psi) \mathrm{d}r + \psi(Z_{t^*}, \infty) \right) \tag{3}$$

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Recall $G(t, u, F, \psi) = \int_0^u \psi(u - x, t) \mathrm{d}F(x) + (1 - F(u)) - \psi(u, t)$.
Then $\Delta_n = \psi_n - \psi_{n-1}$ satisfies

$$\Delta_{n+1}(u,t) = \lambda \mathbb{E}_u \left(\int_t^{t^*} L(r) \int_0^{Z_r} (\Delta_n|_{(Z_r - x, r)} \mathrm{d}F(x) - \Delta_n|_{(Z_r, r)}) \mathrm{d}r \right)$$

Mild Formulation - Continued

The $L^\infty((0,t^*) \times (0,\infty))$ norm of Δ_n can be easily estimated by

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Remark. Energy type estimates can be used to show uniqueness of solutions.

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Theorem (L-T) Assume that $L(t) = 0, \forall 0 \leq t < t^*$, $L(t) = 1 \forall t \geq t^*$ and that $\sigma^2/2 > a$. Then $\psi(u, t) \equiv 1$.

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Note, $\phi(u, t) = 1 - \psi(u, t)$ satisfies on $0 < t < t^*$,

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Use maximum principle developed by Cosner (1980) to show that $\phi(u, t) = 0 \ \forall 0 \leq u$ and $\forall 0 < t < t^*$ as claimed.

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- Analyze examples with simple distributions for X to understand effect of delay in asymptotic behavior

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