The Distribution of The Total Dividend Payments in a MAP Risk Model with Multi-Threshold Dividend Strategy

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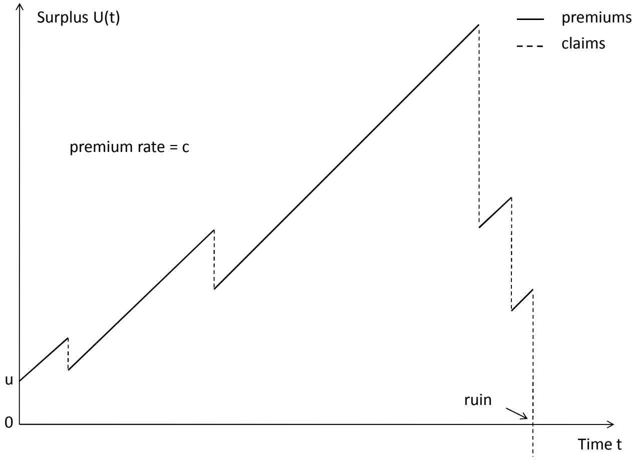
Outline of Topics

Sample Surplus Process



The Classical Risk Model

- The surplus process $\{U(t); t \geq 0\}$ with $U(0) = u$, s.t.

$$dU(t) = cdt - dS(t), \quad t \geq 0.$$

- Premiums are collected continuously at a constant rate c
- A sequence of non-negative claim amounts r.v.$\{X_n; n \in \mathbb{N}^+\}$
- Number of claims up to time t, $N(t) \sim \text{Poisson}(\lambda t)$
- Aggregate claim amounts up to time t, $S(t) = \sum_{n=1}^{N(t)} X_n$
- Time of ruin $\tau = \inf\{t \geq 0 : U(t) < 0\}$

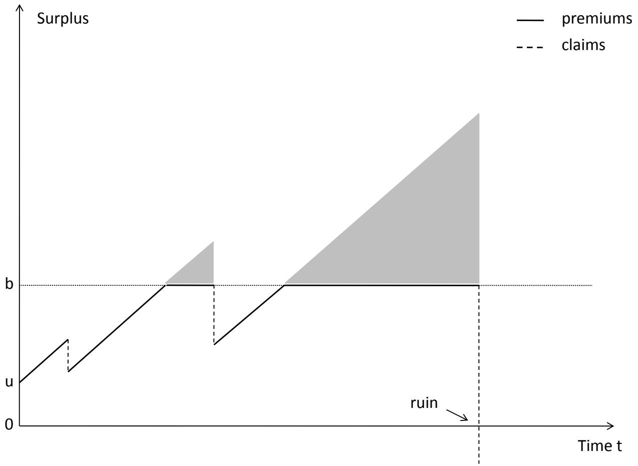
MAP Risk Model

MAP $(\vec{\alpha}, \mathbf{D_0}, \mathbf{D_1})$

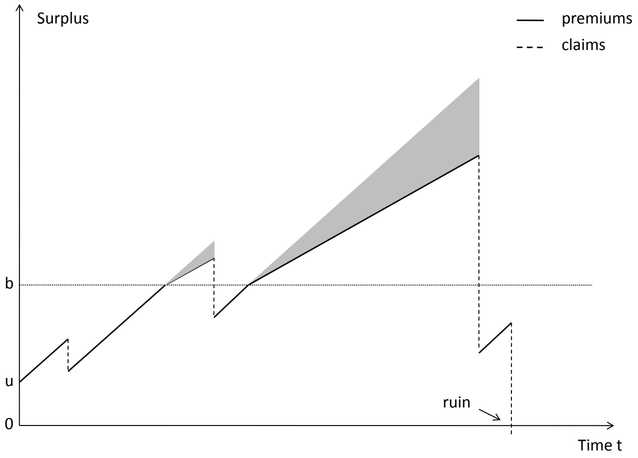
- Initial distribution, $\vec{\alpha}$
- Intensity matrix, $\mathbf{D_0} + \mathbf{D_1}$
- Intensity of state changing without claim, $D_0(i,j) \geq 0$, $j \neq i$
- Intensity of state changing with claim, $D_1(i,j) \geq 0$
- The diagonal elements of $\mathbf{D_0}$ are negative values, s.t. $\mathbf{D_0} + \mathbf{D_1} = \mathbf{0}$
- Special cases: classical risk model, Sparre-Andersen risk model, Markov-modulated risk model

Reference: Badescu et al. (2007), Badescu (2008), Ren (2009),

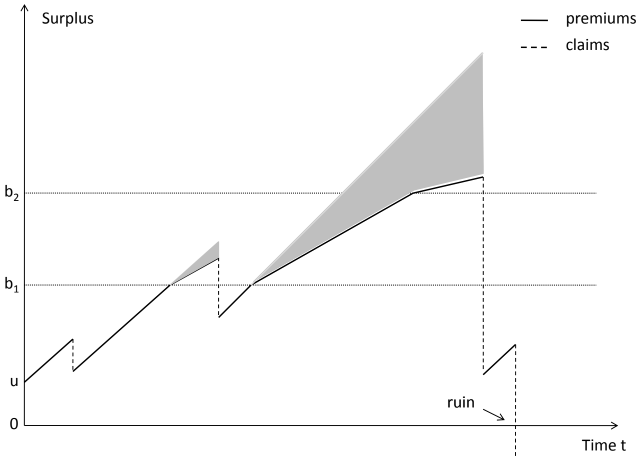
Various Dividend Strategies



Various Dividend Strategies



Various Dividend Strategies



Multi-Threshold MAP Risk Model

- Thresholds: $0 = b_0 < b_1 < \cdots < b_n < b_{n+1} = \infty$
- Premium rate c_k for $b_{k-1} \leq u < b_k$, $k = 1, \cdots, n+1$
  $c = c_1 > c_2 > \cdots > c_n > c_{n+1} \geq 0$
- Time of ruin $\tau_B = \inf\{t \geq 0 : U_B(t) < 0\}$
- Surplus process $\{U_B(t); t \geq 0\}$ satisfies

$$dU_B(t) = c_k dt - dS(t), \quad b_{k-1} \leq U_B(t) < b_k$$

- Claim amounts distribution $f_{i,j}$, $F_{i,j}$ and Laplace transformation $\hat{f}_{i,j}(s)$

Expected Discounted Dividend Payments

- $D(t)$ is the aggregate dividends paid by time t
- Define

$$D_{u,B} = \int_0^{\tau_B} e^{-\delta t} dD(t), \quad u \geq 0,$$

  to be the present value of dividend payments prior to ruin, given the initial surplus u
- Define

$$V_i(u; B) = \mathbb{E}_i[D_{u,B} | U_B(0) = u], \quad i \in E,$$

  to be the expected present value of dividend payments prior to ruin, given the initial surplus u and the initial phase $i \in E$

Expected Discounted Dividend Payments

- The piecewise vector function of the expected present value of the total dividend payments prior to ruin

$$\vec{V}(u;B) = \begin{cases} \vec{V}_1(u;B) & 0 \le u < b_1, \\ \vec{V}_k(u;B) & b_{k-1} \le u < b_k, \quad k = 2, \cdots, n, \\ \vec{V}_{n+1}(u;B) & b_n \le u < \infty. \end{cases}$$

- $\vec{V}_k(u;B) = (V_{1,k}(u;B), \cdots, V_{m,k}(u;B))^\top$ for $b_{k-1} \le u < b_k$ and $k = 1, \cdots, n+1$

Differential Approach

- Typical approach in various risk models
- Integro-differential equations are involved
- Can be derived and solved analytically for some families of claim amounts distribution
- Mainly in Gerber-Shiu discounted penalty function
  Techniques can be applied to the dividend payments problems
- Lin and Sendova (2008), classical risk model
  Lu and Li (2009), Sparre Andersen risk model

Integro-Differential Equation for $\vec{V}_k(u; B)$

- Condition on the events occurring in a small time interval $[0, h]$
  - No change in the MAP state
  - A change in the MAP state accompanied by no claim arrival
  - A change in the MAP state accompanied by a claim arrival; Claim amounts may vary
  - Two or more events occur

Integro-Differential Equation for $\vec{V}_k(u; B)$

- Integro-differential equation, for $b_{k-1} \leq u < b_k$

$$c_k \vec{V}_k'(u; B) = \delta \vec{V}_k(u; B) - \mathbf{D_0} \vec{V}_k(u; B) - \int_0^{u-b_{k-1}} \mathbf{\Lambda_f}(x) \vec{V}_k(u-x; B) dx - \vec{\gamma}_k(u)$$

where $\gamma_{i,k}(u) = (c - c_k) + \sum_{j=1}^{m} D_1(i,j) \sum_{l=1}^{k-1} \int_{u-b_l}^{u-b_{l-1}} V_{j,l}(u-x; B) dF_{i,j}(x)$

- Solution

$$\vec{V}_k(u; B) = \mathbf{v}_k(u - b_{k-1}) \vec{V}_k(b_{k-1}; B) - \frac{1}{c_k} \int_0^{u-b_{k-1}} \mathbf{v}_k(t) \vec{\gamma}_k(u-t) dt$$

where $\mathbf{v}_k(u - b_{k-1}) = \mathcal{L}^{-1} \left\{ \left[\left(s - \frac{\delta}{c_k} \right) \mathbf{I} + \frac{1}{c_k} (\mathbf{D_0} + \mathbf{\Lambda_{\hat{f}}}(s)) \right]^{-1} \right\}$

Recursive Expression for $\vec{V}_k(u; B)$

- Define vector function $\vec{V}_k(u)$ for $u \geq b_{k-1}$

$$\vec{V}_k(u) = \mathbf{v}_k(u - b_{k-1})\vec{V}_k(b_{k-1}) - \frac{1}{c_k}\int_0^{u-b_{k-1}} \mathbf{v}_k(t)\vec{\gamma}_k(u-t)dt$$

- Restrict to $b_{k-1} \leq u < b_k$, compare with $\vec{V}_k(u; B)$

$$\vec{V}_k(u; B) = \vec{V}_k(u) + \mathbf{v}_k(u - b_{k-1})\vec{\pi}_k(B), \quad b_{k-1} \leq u < b_k$$

- Continuity condition at b_{k-1}, $k = 1, \cdots, n$

$$\vec{\pi}_{k+1}(B) = \vec{V}_k(b_k) - \vec{V}_{k+1}(b_k) + \mathbf{v}_k(b_k - b_{k-1})\vec{\pi}_k(B)$$

- Final boundary condition when $k = n + 1$

$$\vec{\pi}_{n+1}(B) = \vec{V}_n(b_n) - \vec{V}_{n+1}(b_n) + \mathbf{v}_n(b_n - b_{n-1})\vec{\pi}_n(B) = \vec{0}$$

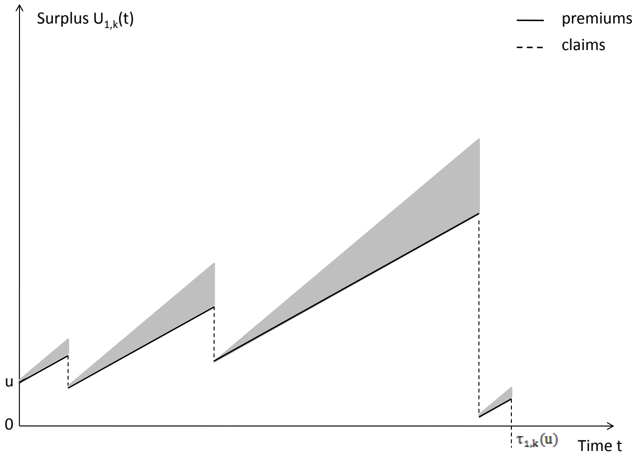
Layer-Based Recursive Algorithm

- Computational disadvantage of the recursive algorithm based on integro-differential equations
  - Constant vectors can only be solved in the last layer
  - Infeasible to compute for large number of layers
- Layer-based approach
  - Condition on the exit times of the surplus out of each layer
  - Calculate successively for increasing number of layers

$$\text{The } k\text{-layer model} \Leftarrow \begin{cases} \text{The } (k-1)\text{-layer model} \\ \text{Classical one-layer model} \end{cases}$$

Reference: Albrecher and Hartinger (2007)

Sample Path of One-Layer Model with Dividend Payments



Time Value of Upper Exit

- Define $\tau^*(u,a,b) = \inf\{t \geq 0 : U(t) \notin [a,b] | U(0) = u\}$
- Define

$$\tau^+(u,a,b) = \begin{cases} \tau^*(u,a,b) & \text{if } U(\tau^*(u,a,b)) = b \\ \infty & \text{if } U(\tau^*(u,a,b)) < a \end{cases}$$

  and

$$\tau^-(u,a,b) = \begin{cases} \infty & \text{if } U(\tau^*(u,a,b)) = b \\ \tau^*(u,a,b) & \text{if } U(\tau^*(u,a,b)) < a \end{cases}$$

- Laplace transform of $\tau_k^+(u,0,b)$

$$B_{i,j,k}(u,b) = \mathbb{E}\left[e^{-\delta\tau_k^+(u,0,b)}\mathbf{1}_{[J(\tau_k^+(u,0,b))=j]}\middle|J(0)=i\right]$$

  given initial phase i and reaching b in phase j

Reference: Gerber and Shiu (1998), Albrecher and Hartinger (2007)

Time Value of Upper Exit

For $\delta > 0$ and $k \in \mathbb{N}^+$, we have

1.
$$\begin{aligned} \mathbf{B}_k &= \mathbf{1}, \quad \text{if } u \geq b \\ \mathbf{B}_k &= \mathbf{0}, \quad \text{if } u < 0 \end{aligned}$$

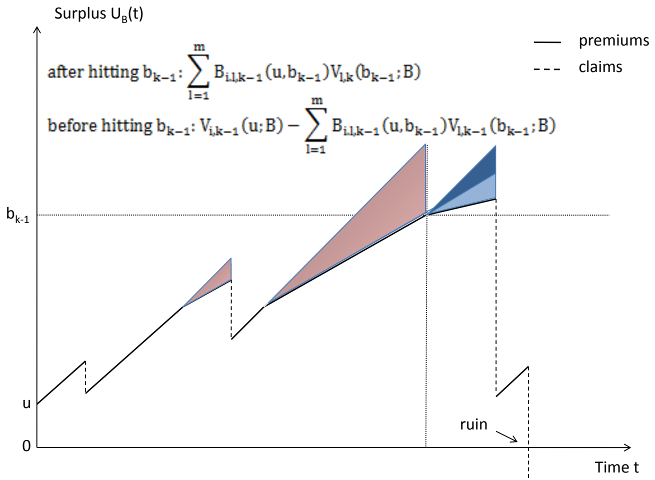
2. For $0 \leq u < b_{k-1}$
$$\mathbf{B}_k(u,b) = \begin{cases} \mathbf{B}_{k-1}(u,b), & \text{if } b \leq b_{k-1} \\ \mathbf{B}_{k-1}(u,b_{k-1})\mathbf{B}_k(b_{k-1},b), & \text{if } b \geq b_{k-1} \end{cases}$$

3. For $b_{k-1} \leq u \leq b$
$$\begin{aligned} \mathbf{B}_k(u,b) &= \mathbf{B}_{1,k}(u-b_{k-1}, b-b_{k-1}) + \mathbf{M}_k(u-b_{k-1}) \\ &\quad -\mathbf{B}_{1,k}(u-b_{k-1}, b-b_{k-1})\mathbf{M}_k(b-b_{k-1}) \end{aligned}$$

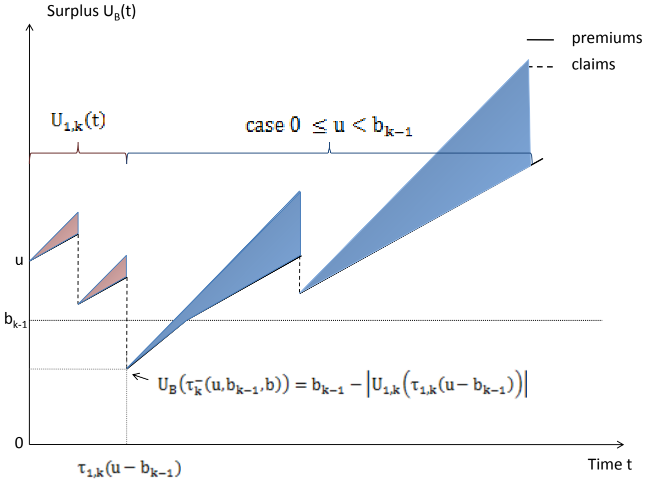
- Parallel results in matrix form

Reference: Albrecher and Hartinger (2007)

Sample Path for $0 \leq u \leq b_{k-1}$



Sample Path for $u \geq b_{k-1}$



Expected Discounted Dividend Payments

- For $0 \leq u \leq b_{k-1}$

$$\vec{V}_k(u;B) = \vec{V}_{k-1}(u;B) + \mathbf{B}_{k-1}(u,b_{k-1})\left[\vec{V}_k(b_{k-1};B) - \vec{V}_{k-1}(b_{k-1};B)\right]$$

- For $u \geq b_{k-1}$

$$\begin{aligned} & \vec{V}_k(u;B) \\ = \quad & \vec{V}_{1,k}(u-b_{k-1}) + \mathbb{E}\left[e^{-\delta\tau_{1,k}(u-b_{k-1})}\vec{V}_k(b_{k-1} - |U_{1,k}(\tau_{1,k}(u-b_{k-1}))|;B)\right] \end{aligned}$$

"Contagion" Example

- State A: standard claims, $\lambda_1 = 1$, $1/\beta_1 = 1/5$
- State B: additional infectious claims, $\lambda_2 = 10$, $1/\beta_2 = 3$
- State A $\rightarrow$ B, $\alpha_A = 0.02$; State B $\rightarrow$ A, $\alpha_B = 1$
- $\mathbf{D_1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 + \lambda_2 \end{pmatrix}$, $\mathbf{D_0} = \begin{pmatrix} -\alpha_A - \lambda_1 & \alpha_A \\ \alpha_B & -\alpha_B - \lambda_1 - \lambda_2 \end{pmatrix}$
- Thresholds $(0, 20, 40, \infty)$, premium rates $(2, 1.5, 1)$

u	$\delta = 0.1$	$\delta = 0.01$	$\delta = 0.001$	Badescu et al. (2007)
0	158.99	323.23	356.68	N/A
10	350.55	457.58	500.95	503.00
30	417.19	671.02	692.82	692.60
50	688.25	802.29	821.50	842.07
70	814.98	926.93	942.78	968.82

Conclusion

- Differential approach is applicable to the MAP risk model
- Moment generating function and higher moments
- Layer-based approach provides an alternative method

Reference

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