Portfolio optimization with a GMMB and risk-adjusted fees

Anne MacKay
(UQAM, Université de Sherbrooke)

joint work with Adriana Ocejo (UNC Charlotte)

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Motivation

- Variable annuity: investment product for retirement savings
  - Financial guarantees during the accumulation and payout phase
  - Also include life insurance benefits
- Investment mix typically pre-determined, static.
- Guaranteed minimum maturity benefit
  - Payoff: max(account value, guaranteed amount)
  - Put option on account value
- Financial guarantee financed by a fee from the investment account.
  - Fee rate set such that the value of the VA is fair (from a risk-neutral perspective)

Motivation

Given a fee structure, what dynamic investment mix will be the most attractive for a policyholder?

- Variation of Merton's portfolio problem
- Non-concave utility:
  - Financial guarantee: utility is non-concave in the terminal wealth (Carpenter, 2000; Chen, Hieber, and Nguyen, 2019)
  - S-shaped utility (Kahneman and Tversky, 1979, 1986)
- Guarantee fee:
  - Affects returns
  - Total rate depends on investment mix
  - Investment strategy is no longer self-financing
- Fair pricing constraint

Some literature

Related work on (constrained) non-concave utility maximization:

- Carpenter (2000):
  - Manager compensation (unconstrained) problem.
- Chen, Hieber, and Nguyen (2019):
  - Hybrid investment-insurance contract
  - No fees
- He and Kou (2018), Dong and Zheng (2020), Nguyen and Stadje (2020):
  - S-shaped utility
  - Constraints.

Impact of guarantee fee

- Fee rate reduces return, affects the value of the guarantee
  $\Rightarrow$ Impacts policyholder behaviour (M. et al., 2017)

- If a dynamic investment mix is allowed...
  - How should the guarantee fee be set up?
  - How will the fee rate(s) affect the optimal investment strategy?
  - Is there an optimal way to set up the fee structure?

Setting

- Policyholder can invest in a risky asset S and a risk-free asset P:

$$dS_t = \mu S_t \, dt + \sigma S_t dW_t$$
$$dP_t = r P_t \, dt$$

- VA account value process built by investing the proportion π_t in the risky asset and deducting the guarantee fee:

$$dF_t = \pi_t F_t \frac{dS_t}{S_t} + (1 - \pi_t) F_t \frac{dP_t}{P_t} - dC_t$$

- GMMB rider guarantees amount G at maturity T:

$$\text{Payoff:} \qquad \max(F_T, G), \qquad \text{with } G = F_0 e^{gT}.$$

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Dynamics of VA account F

- Consider two levels of fee:
  - c_F paid on the total value of the account, and
  - Additional fee $c_S < \mu - r$ paid on the risky investment.
- Accumulated fees up to t follow:

$$C_t = \int_0^t (c_F + \pi_s c_S) F_s \, ds, \qquad C_0 = 0.$$

- VA account value has dynamics:

$$\begin{aligned}\frac{dF_t}{F_t} &= [\pi_t(\mu - r - c_S) + r - c_F]dt + \pi_t \sigma \, dW_t \\ &= [\pi_t(\widetilde{\mu} - \widetilde{r}) + \widetilde{r}]dt + \pi_t \sigma \, dW_t,\end{aligned}$$

with $\widetilde{\mu} = \mu - c_S - c_F$ and $\widetilde{r} = r - c_F$.

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Admissible investment strategies

$\pi = (\pi_t)_{t \geqslant 0}$ is in the set of admissible trading strategies $\mathcal{A}(x)$ for an initial investment x if:

- π_t is $\mathcal{F}_t$-measurable,
- $F_0^\pi = x$,
- $F_t^\pi \geqslant 0$ a.s.,
- there exists a unique solution to the SDE

$$\frac{dF_t^\pi}{F_t^\pi} = [\pi_t(\widetilde{\mu} - \widetilde{r}) + \widetilde{r}]dt + \pi_t \, \sigma \, dW_t.$$

S-shaped utility function

- Classical utility function: $u(\cdot)$ strictly increasing, strictly concave, and continuously differentiable on $\mathbb{R}^+$ and $u(0) = \lim_{x\downarrow 0} u(x)$.
- S-shape utility function:

$$U(x) = \begin{cases} -U_2(\theta - x), & 0 \leqslant x < \theta, \\ U_1(x - \theta), & x \geqslant \theta, \end{cases}$$

with U_1, U_2 are classical utility functions with

- $U_1(0) = -U_2(0) \geqslant 0$,
- $\lim_{x\uparrow\infty} U_1(x) = \infty$,
- $\lim_{x\to\infty} U_1'(x) = 0$, $\lim_{x\to 0} U_1'(x) = \infty$, $\lim_{x\to\infty} \frac{xU_1'(x)}{U_1(x)} < 1$ (Inada and asymptotic elasticity conditions).

Constrained dynamic portfolio problem

We want to solve

$$\max_{\pi \in \mathcal{A}(F_0)} \mathbb{E}[U(\max(F_T^\pi, G))]$$
$$\text{s.t.} \quad \mathbb{E}[\xi_T \max(F_T^\pi, G)] = F_0,$$

where U is an S-shape utility function and ξ_T is the state-price density.

- $E[\xi_T \max(F_T^\pi, G)] = F_0$ is the fair pricing constraint.
  - Fair pricing depends on the investment strategy π.

Economic interpretation of the problem

- What dynamic investment mix can the VA provider offer if they want to maximize (some) policyholder's utility while keeping the contract fairly priced?
- What does the resulting payoff look like?
- What is the "best" way to set the fees?

Solving the unconstrained problem

Use martingale approach with static optimization problem:

$$\arg\max_{H\in\mathcal{H}} \mathbb{E}[U(\max(H,G))], \quad s.t. \quad \mathbb{E}[\widetilde{\xi}_T H] \leqslant F_0, \tag{1}$$

with $\mathcal{H} = \{H : H \text{ is } \mathcal{F}_T\text{-measurable}, H \geqslant 0 \ \mathbb{P} - a.s.\}$ and where $\widetilde{\xi}_t$ is the state price density corresponding to the "fee-adjusted" market

$$d\widetilde{P}_t = \widetilde{r}\,\widetilde{P}_t\,dt, \qquad d\widetilde{S}_t = \widetilde{S}_t\,(\widetilde{\mu}\,dt + \sigma\,dW_t).$$

- Optimal payoff can always be replicated because the fee-adjusted market is complete.
- Main tool for solving the static problem: concavification of the utility function (Carpenter, 2000; Reichlin, 2013; Bichuch and Sturm, 2014)

Proposition 3.1 of M. and Ocejo (2022)

Let $U(\cdot)$ be an S-shaped utility function and $M := \max(\theta, G)$. The solution to the unconstrained static optimization problem (1) is given by

$$H^* = [I(\lambda\widetilde{\xi}_T) + \theta]\mathbb{1}_{\{\lambda\widetilde{\xi}_T < \hat{y}\}}, \quad (2)$$

where

- $I(x) = (U_1'(x))^{-1}$,
- $\lambda \geqslant 0$ is such that $\mathbb{E}[\widetilde{\xi}_T H^*] = F_0$, and
- $\hat{y} := U_1'(\hat{x} - \theta)$, where $\hat{x} \in (M, \infty)$ is the unique root of the equation

$$U_1(x - \theta) - xU_1'(x - \theta) - U(G) = 0.$$

Solving the constrained problem

1. Static problem:

$$\max_{H \in \mathcal{H}} \mathbb{E}[U(\max\{H, G\})], \qquad \text{s.t.} \qquad \mathbb{E}[\widetilde{\xi}_T H] \leqslant F_0,$$
$$\mathbb{E}[\xi_T \max\{H, G\}] = F_0.$$

2. Representation problem: find $\pi^* \in \mathcal{A}(F_0)$ s.t. $F_T^{\pi^*} = H^*$.

Admissible fees

For fixed maturity T, guaranteed roll-up rate g, define the set of admissible fees

$$\mathcal{P}_{T,g} = \{(c_F, c_S) : \mathbb{E}[\xi_T \max\{H^\star, G\}] \geqslant F_0, \text{ where } H^\star \text{ solves the unconstrained static problem.}\}$$

Lagrangian of the static problem

The Lagrangian of the static problem is given by

$$L(x;\ y,z) := \widetilde{U}(x) - xy - z\max\{x,G\}, \qquad x \geqslant 0,$$

where $\widetilde{U}(x) := U(\max\{x,G\})$,

- $y := \lambda_1\widetilde{\xi}_T$ (from $\mathbb{E}[\widetilde{\xi}_T H^*] = F_0$), and
- $z := \lambda_2\xi_T$ (from $\mathbb{E}[\xi_T \max\{H^*,G\}] = F_0$).

Proposition 3.2 of M. and Ocejo (2022)

For each $y \geqslant 0$ and $z > 0$, the maximizer of the Lagrangian is:

$$\chi(y,z) = \begin{cases} \chi_1(y,z) = [I(y+z)+\theta]\mathbb{1}_{\{\Delta(y+z)+zG>0\}}, & \text{if } \theta \leqslant G, \\ \chi_2(y,z) = [I(y+z)+\theta]\mathbb{1}_{\{0<y+z<U_1'(\hat{x}-\theta)\}}\mathbb{1}_{\{\Delta(y+z)+zG>0\}}, & \text{if } \theta > G, \end{cases}$$

- $\Delta : [0,\infty) \mapsto \mathbb{R}$ is defined by $\Delta(a) := U_1(I(a)) - a[I(a)+\theta] - \widetilde{U}(0)$.
- $\hat{x} \in (\theta, \infty)$ is the unique root of

$$U_1(x-\theta) - (x-G)U_1'(x-\theta) - \widetilde{U}(0) = 0.$$

- In both cases, the maximizer is either larger than $\max(\theta, G)$ or equal to 0.

Idea of the proof

1. Split the Lagrangian in two:

$$\sup_{x \geqslant 0} L(x;\, y, z) = \max\{ \sup_{0 \leqslant x < G} L(x;\, y, z), \sup_{x \geqslant G} L(x;\, y, z)\}.$$

2. For $x \in [0, G)$, $L(0;\, y, z) = \widetilde{U}(0) - zG$ is the supremum.

3. For $x \in [G, \infty)$, write $w := x - G$ and $V(w) := \widetilde{U}(w + G)$.
   - If $\theta \leqslant G$, $V(w)$ is concave $\Rightarrow$ use first-order condition.
   - If $\theta > G$, $V(w)$ is not concave $\Rightarrow$ use concavification techniques as in the unconstrained problem.

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Theorem 3.1 of M. and Ocejo (2022)

For $(c_F, c_S) \in \mathcal{P}_{T,g}$, the solution to the constrained static problem is given by

$$H^* = \chi(\lambda_1 \widetilde{\xi}_T, \lambda_2 \xi_T),$$

where $\lambda_1 \geqslant 0$, $\lambda_2 > 0$ are such that

$$\mathbb{E}[\xi_T \max(H^*, G)] = F_0$$

and either $\lambda_1 = 0$ or $\mathbb{E}[\widetilde{\xi}_T H^*] = F_0$.

Existence of multipliers

- Proof follows Chen, Hieber, and Nguyen (2019).
- Can split the fee rate vectors in 3 categories:
  1. Fee rates are not in $\mathcal{P}_{T,g}$, utility is maximized by the solution to the unconstrained problem;
  2. $\lambda_1^\star = 0$: budget constraint is not binding, contract is fair, no solution to the representation problem;
  3. $\lambda_1^\star, \lambda_2^\star > 0$: both constraints are binding and the constrained dynamic portfolio problem has a solution.

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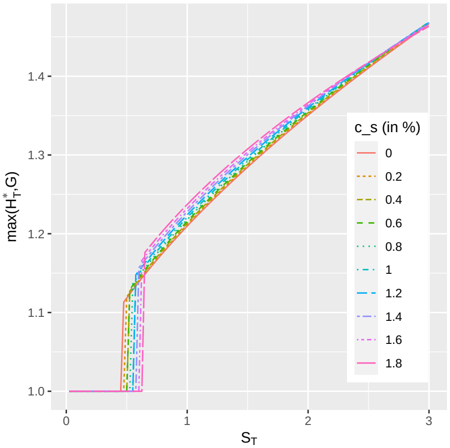
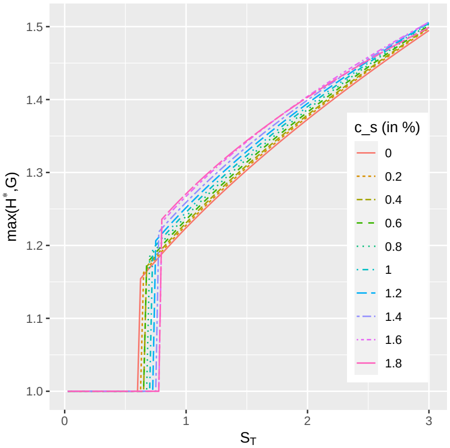
Numerical illustration

- Variable annuity contract with $T = 10$, $F_0 = G = 1$.
- Market parameters: $\mu = 0.04$, $r = 0.02$, $\sigma = 0.2$, $S_0 = 1$.
- $U_i(x) = x_i^{\gamma}/\gamma_i$, $\gamma_1 = 0.2$, $\gamma_2 = 0.4$.

▶ Some remarks:
  - Fair fee rate if $\pi_t \equiv 1$: $c_F + c_S = 2.45\%$.
  - Constrained dynamic portfolio problem has a solution for all fee rates considered.

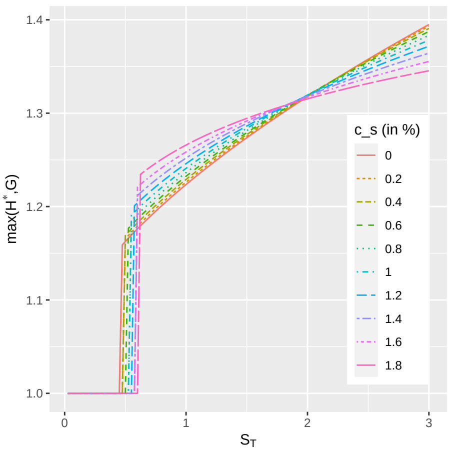
Impact of c_S on optimal payoff, $\theta < G$

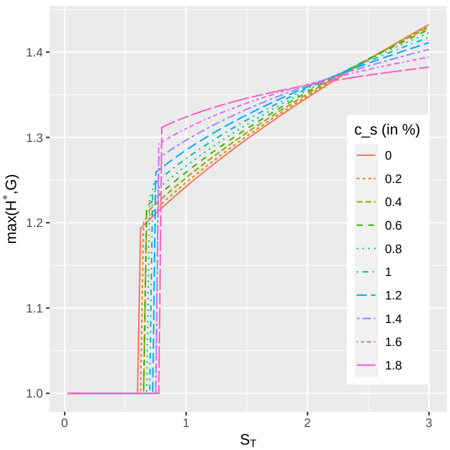




Optimal payout $\max(H^*, G)$ as a function of S_T, $\theta = 0.95F_0$ (left: $c_F = 1\%$, right: $c_F = 2\%$)

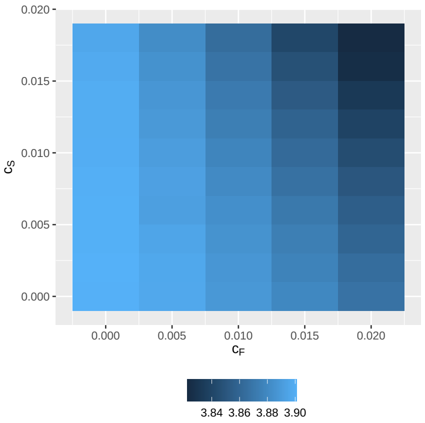
Impact of c_S on optimal payoff, $\theta > G$



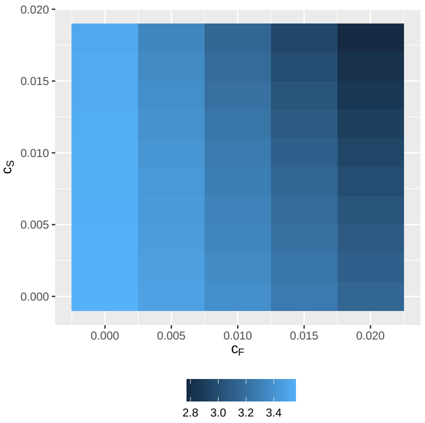


Optimal payout $\max(H^*, G)$ as a function of S_T, $\theta = 1.05F_0$ (left: $c_F = 1\%$, right: $c_F = 2\%$)

Optimal expected utility $E[U(\max(H^*, G))]$

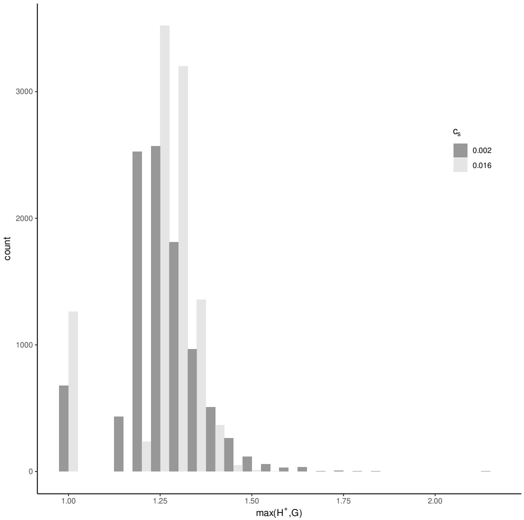


$\theta = 0.95 F_0$

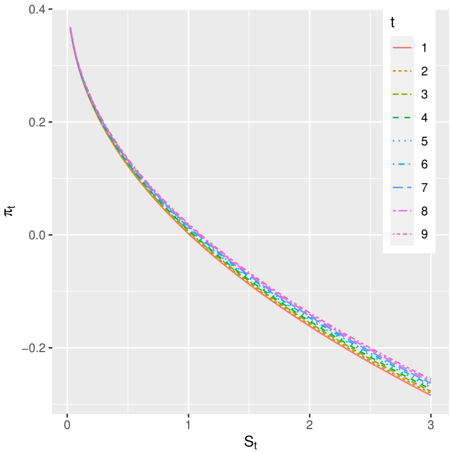


$\theta = 1.05 F_0$

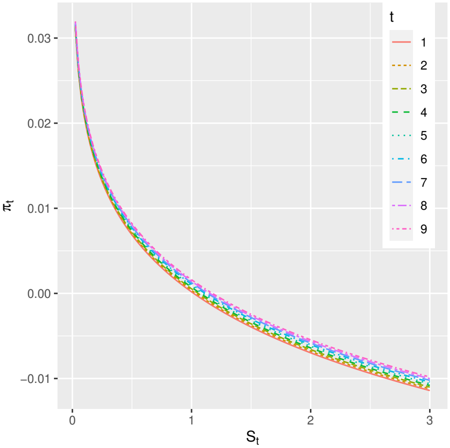
Impact of c_S on distribution of payoff



Impact of c_S on the optimal investment strategy

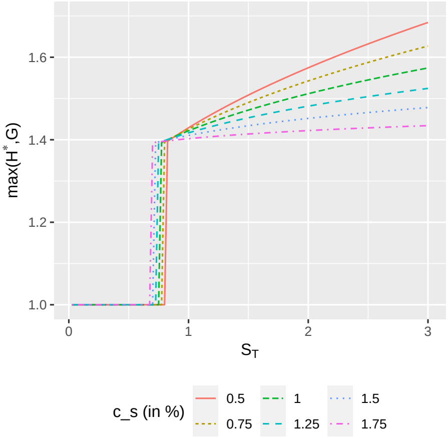


$c_S = 0.2\%$

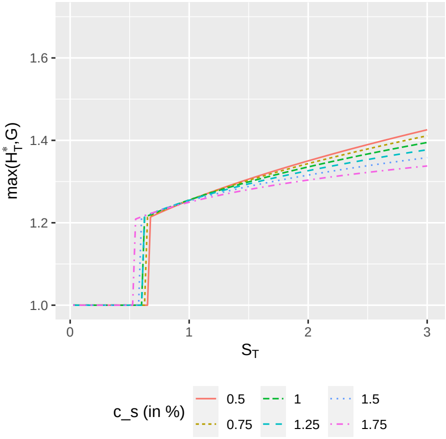


$c_S = 1.6\%$

Unconstrained vs constrained optimal payout



Unconstrained



Constrained

$$\theta = 1.05F_0, \; c_F = 0.02448 - c_S$$

Concluding remarks

- Constrained, non-concave utility maximization.
- Use of auxiliary market to account for fee outflow.
- Utility of policyholder maximized with lower fees (linked to more conservative payouts).
- VA products maximize policyholder's expected utility by offering dynamic investment strategies, especially if fees are low.

Thank you for your attention!

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