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Tight Approximation Of Basic Characteristics Of Classical And Non-Classical Surplus Processes*

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Abstract

We propose asymptotically correct two-sided bounds for random sums (where the number of summands has an arbitrary distribution) which can be viewed as ruin probabilities or accumulated claim sizes in various risk processes. The bounds are fairly new and they reveal the real accuracy of some well-known and widely used asymptotic formulas. It turns out that this accuracy can be poor and the range where these formulas give appropriate approximations depends on the shape of the tail distribution of the summands. This dependence is examined in the research. We also propose some routines which can be used in actuarial practice. They give bounds of the probability of ruin which are fairly close to the real ruin probability.

The report consists of three sections. Section 1 gives a general oversight of the results obtained during the research. Section 2 is designed for readers interested in mathematical aspects of the work. Section 3 contains numerical routines and corresponding numerical results illustrating our approach.

Key words: characterization, geometric sum, reserve, ruin probability, severity, subexponential distribution, two-sided bounds.

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1 Scope of the research and the results obtained

1.1 Risk models

Actuarial science deals with the random level of risk, depending on the frequency and the severity of claims, to be covered by the insurer (see Panjer and Willmot [32]). In order to study the insurer's risk, various *risk models* were proposed (see [1, 2, 9, 12, 32]). All these models use various assumptions concerning the claim occurrence process, income (premium) process, severities distributions, etc., but all they incorporate the following characteristics which are of special importance for the insurer.

(i) The *accumulated claims*

$$C_t = \sum_{k \le N_t} Z_k \tag{1.1}$$

for an accounting period t, where N_t is the total number of claims that occurred before time t, and Z_k is the kth claim size.

(ii) The insurer's *surplus* $R(t)$ at time t which is defined as "the excess of the income (with respect to portfolio of business) over the outgo, or claims paid", see [32, p. 357]. This characteristic reflects the *solvency* of the insurer.

(iii) The *probability of ruin*

$$\Psi(x) = \mathbf{P}\left(\inf_t R(t) < 0 \mid R(0) = x\right), \quad x \ge 0, \tag{1.2}$$

which is the probability that the insurer's surplus reaches a fixed minimal level (taken as 0 for simplicity) provided that the initial reserve (or capital) is x. The probability of ruin reflects the volatility inherent in the insurance business, and it "serves as a useful tool in long range planning for the use of an insurer's funds" (see [32, p. 357]). It is possible to say that the ruin probability measures the risk level of the insurer having the initial capital x.

In practice, we are interested in the probability $\mathbf{P}(C_t > x)$ that the accumulated claims exceed a *high level* x. In other words, we are interested in asymptotic behavior of the right tail of the accumulated claims distribution. Similarly, we are interested in *small values* of ruin probability $\Psi(x)$ which corresponds to the case of large initial capital x. These characteristics represent the principal interest for the risk theory.

Mathematical analysis of both the distribution of the accumulated claims and the probability of ruin is quite similar. Because of this, let us explain the problems solved and the results obtained in the research taking the probability of ruin as the illustration.

As we have noted, only small values of the ruin probability corresponding to large values of the initial reserve are of practical interest. The following dual problems are usually under consideration in risk theory.

(i) To find the risk level (measured as the value of the probability of ruin $\Psi(x)$) provided that the initial capital x of the insurer is fixed;

(ii) Given the risk level Ψ^* to find the initial capital x securing the ruin probability below the prescribed level Ψ^* that is, $\Psi(x) \leq \Psi^*$.

Both these problems can easily be solved if the exact expression of the ruin probability is known. But this is not the case for even rather simple models. Because of this, it is necessary to use various approximations of ruin probabilities. This necessity is well-known, it has been discussed widely, and a variety of approximations has been proposed, see [1, 2, 6, 8, 10, 12, 13, 15, 16, 20, 32, 35, 36].

1.2 Two asymptotic approximations of ruin probabilities

Let us mention two famous approximations which are valid for the so-called *classical risk model* in which claims occur in accordance with the Poisson process and claim sizes (severities) are independent and identically distributed (i.i.d.). If the claims are "small" (that is, their moment generating function is finite – this is true, for example, for exponential, mixed exponential, gamma, and other distributions), then, typically, the following *Cramér-Lundberg asymptotic formula* is valid

$$\Psi(x) \sim \Psi_{CL}(x) = k_{CL} \exp(-\varepsilon_{CL} x), \tag{1.3}$$

where $\varepsilon_{CL} > 0$ is the Cramér exponent called in actuarial practice the *adjustment coefficient*, and k_{CL} is the Cramér-Lundberg constant. The notation $\Psi(x) \sim \Psi_{CL}(x)$ means that

$$\frac{\Psi(x)}{\Psi_{CL}(x)} \to 1 \quad \text{as } x \to \infty.$$

In this case, the ruin typically results from the accumulation of relatively small claims and formula (1.3) shows that the probability of ruin decays asymptotically as an exponential function when the initial capital tends to infinity. Approximation (1.3) is discussed in many sources, see, e.g., [9, 10, 12, 13, 15, 16].

The second approximation refers to the case of so-called large claims with severities having Pareto, Weibull, lognormal, loggamma, and other distributions. These distributions emerge when one models claims occurring from damages caused by hurricanes, tornados, earthquakes, floods, fires, etc., and they are called *subexponential* because they have no finite exponential moment (their moment generating function is infinite); see [9] and Section 2 for more precise definitions. Denote by $B(u)$ the severity distribution function (d.f.) and let b_1 be the mean severity value. In the presence of large claims, the ruin probability typically results from only one extremely large claim that is, the mechanism of ruin differs dramatically from the case of small claims. In the case of large claims the probability of ruin has another asymptotic representation which is

$$\Psi(x) \sim \Psi_{SE}(x) = \frac{1}{\kappa b_1} \int_x^\infty (1 - B(u))\, du, \tag{1.4}$$

where κ stands for the *relative safety loading*, see [9, 10]. One can see that the asymptotic approximation (1.4) is proportional to the *integrated tail* of the severity distribution and it decreases slower than any exponential function. For instance, if B is the Pareto distribution, this expression decays as a power function.

As we have mentioned, the exact values of the probability of ruin can rarely be obtained analytically. But, for large initial reserves, we can use the known asymptotic approximations instead of unknown exact formulas: (1.3) in the case of small claims and (1.4) in the case of large claims. The idea is quite reasonable. However, immediate implementation of approximations (1.3) and, especially, (1.4) can cause serious practical problems which actually inspired this research and which are discussed in the following subsections.

1.3 Accuracy problem as the motivation of the research

Let us assume, for a while, that we use approximations (1.3) and (1.4) instead of the unknown exact formulas. Both these approximations formally mean that either the probability of ruin decays exponentially fast (in the case of small claims), or it is proportional to the integrated tail of the claim size distribution (in the case of large claims). But, strictly speaking, both these relations say nothing about their convergence rates. For example, if one wants to be sure that the error of the approximation (for definitness, of Ψ_{SE}) is within 10% of the exact value Ψ, then it is necessary to be sure that, for a prescribed initial reserve x,

$$0.9\,\Psi_{SE}(x) \le \Psi(x) \le 1.1\,\Psi_{SE}(x).$$

But the relation (1.4) (as well as (1.3)) does not provide this information. In such a situation, the asymptotic approximations are, to say the least, practically useless and, to say more, can be misleading in actuarial practice. This discussion clears up the necessity to examine the accuracy of the asymptotic approximations.

During the last decade, a lot of attention has been paid to the accuracy problem of the Cramér-Lundberg approximation. Let us refer to [12, 13, 15], where the reader can find further references. Our previous research [16] (supported by CKER) was partly devoted to this problem and the basic conclusion was that the Cramér-Lundberg formula has nice accuracy and therefore, the asymptotic approximation (1.3) can be used in actuarial calculations without fear to make large errors.

In this research we deal with large claims where the situation is not so optimistic. So far, the accuracy problem of formula (1.4) has not been examined thoroughly. Only a few works (see [15, 20, 35, 36] where a few additional references can be found) were devoted to bounding of ruin probabilities in the presence of large claims. In our previous research [16] (see also [20]) we showed that the approximation (1.4) can lay far away from real values of ruin probabilities. Just to give an impression, let us consider the case where the severity has the translated Pareto distribution of the form

$$B(u) = 1 - \left(1 + \frac{u}{2}\right)^{-4}$$

and the relative safety loading $\kappa = 0.01$. Then the approximation (1.4) gives

$$\Psi_{SE}(600) \approx 3.7 \cdot 10^{-6},$$

whereas the real value $\Psi(600)$ lies in the interval $(8.9 \cdot 10^{-4}, 6.6 \cdot 10^{-3})$. The relative error of the approximation (1.4) is more than 20000% ! Evidently, this level of "accuracy" is out of common sense and, in this case, the approximation (1.4) cannot be recommended for practical use.

Such exciting facts inspired this research which is partly devoted to the accuracy problem of the approximation (1.4) and also to the related approximation of the accumulated claim size distribution.

In order to state it more precisely, let us say that the approximation Ψ_{SE} has a *relative accuracy* $\delta = \delta(x)$ at point x if the unknown genuine ruin probability $\Psi(x)$ satisfies the inequalities

$$(1 - \delta)\,\Psi_{SE}(x) \le \Psi(x) \le (1 + \delta)\,\Psi_{SE}(x). \tag{1.5}$$

This research is intended to solve the following two closely related problems:

(i) For a prescribed relative accuracy level δ^* (e.g., $\delta^* = 0.1$ corresponds to the 10%-error), find the value x^* of the initial capital above which all values $\delta(x)$ do not exceed δ^* that is,

$$\delta(x) \le \delta^* \quad \text{for all } x \ge x^*. \tag{1.6}$$

If the relation (1.6) holds, then the application of the asymptotic approximation (1.4) leads to a relative error which is not greater than δ^* provided that the initial capital x is greater than or equal to x^*.

(ii) For a prescribed level x^* of the initial capital, find the value δ^* of the relative accuracy such that (1.6) holds. This gives us the opportunity to judge whether the approximation (1.4) is workable for the given level x^* of the initial reserve, or not.

An additional requirement of practical importance is the computability of the relevant quantities.

Another problem consists of the following: "If the approximation $\Psi_{SE}(x)$ is not acceptable, then how can we approximate the unknown ruin probability?" We also deal with this problem and propose some bounds which are not asymptotically correct, in general, but give reasonable accuracy for desired values of the initial reserve and can easily be calculated by computer routines.

The following subsections contain an outline of our approach and explain its basic features.

1.4 Asymptotically correct bounds and their usage

Let us assume that we constructed functions (two-sided bounds) $\Psi^-(x)$ and $\Psi^+(x)$ such that they embrace both the unknown genuine ruin probability and its known asymptotic approximation that is,

$$\Psi^-(x) \le \Psi(x) \le \Psi^+(x), \tag{1.7}$$

$$\Psi^-(x) \le \Psi_{SE}(x) \le \Psi^+(x). \tag{1.8}$$

Assume additionally that these two-sided bounds are asymptotically correct in the sense that

$$\frac{\Psi^+(x)}{\Psi^-(x)} \to 1 \quad \text{as } x \to \infty.$$

It follows, in particular, that

$$\Psi^-(x) \sim \Psi^+(x) \sim \Psi(x) \sim \Psi_{SE}(x). \tag{1.9}$$

In subsection 2.4 we shall show how to construct such bounds for the probability of ruin and, in subsection 2.6, how to generalize the construction to the case of accumulated claim size. Some hints to this construction will be also given in subsection 1.5. The construction itself requires sophisticated mathematical results and the reader interested in the details is referred to Section 2.

We explain now how the bounds Ψ^- and Ψ^+ satisfying (1.7) to (1.9) can be used to answer the questions stated at the end of the preceding subsection. Define the following relative deviations of the bounds from the known approximation Ψ_{SE}:

$$\Delta^-(x) = \frac{\Psi_{SE}(x) - \Psi^-(x)}{\Psi_{SE}(x)}, \tag{1.10}$$

$$\Delta^+(x) = \frac{\Psi^+(x) - \Psi_{SE}(x)}{\Psi_{SE}(x)}, \tag{1.11}$$

Since we know all the quantities Ψ^-, Ψ^+, and Ψ_{SE}, we can easily find both Δ^- and Δ^+. Using (1.9),

$$\Delta^-(x) \to 0 \quad \text{as } x \to \infty$$

and

$$\Delta^+(x) \to 0 \quad \text{as } x \to \infty.$$

Furthermore, by (1.10) and (1.11),

$$\Psi^-(x) = \Psi_{SE}(x)\left(1 - \Delta^-(x)\right), \tag{1.12}$$

$$\Psi^+(x) = \Psi_{SE}(x)\left(1 + \Delta^-(x)\right). \tag{1.13}$$

Therefore, by (1.7), we have the following accuracy estimate

$$\left(1 - \Delta^-(x)\right)\ \Psi_{SE}(x) \leq \Psi(x) \leq \left(1 + \Delta^+(x)\right)\ \Psi_{SE}(x)$$

which can easily be reduced to (1.5) by taking

$$\delta(x) = \max\left(\Delta^-(x), \Delta^+(x)\right).$$

As both $\Delta^-(x)$ and $\Delta^+(x)$ tend to 0 when $x \to \infty$, the value $\delta(x)$ defined above also tends to 0 as $x \to \infty$. This makes it possible for us to solve the problems posed at the end of the preceding subsection. Namely, let δ^* be a prescribed relative accuracy of the approximation Ψ_{SE}. Take

$$x^* = \inf\{x : \delta(u) \leq \delta^* \text{ for all } u \geq x\}. \tag{1.14}$$

This value x^* is the minimal level of the initial capital above which we can use the approximation Ψ_{SE} with the given accuracy. Note that, typically, $\delta(x)$ is a monotonically decreasing function and, in such a case, we can find x^* from a simpler equation:

$$x^* = \inf\{x : \delta(x) \leq \delta^*\}. \tag{1.15}$$

Similarly, if we are given the required level of the initial capital x^*, then the accuracy of the approximation Ψ_{SE} is equal to $\delta^* = \delta(x^*)$.

1.5 Mathematics behind the asymptotically correct bounds

In this subsection we outline how to find the quantities Δ^- and Δ^+ defined in (1.10) and (1.11). We do not give all relevant details here (the reader is referred to subsections 2.2 to 2.6 for the details) and we restrict ourselves to the classical risk model and to the corresponding ruin probability which has the asymptotic representation (1.4) in the case where the severity d.f. B is such that the auxiliary d.f.

$$F(u) = \frac{1}{b_1}\int_0^u (1 - B(z))\, dz \tag{1.16}$$

is subexponential (see subsection 2.2 for exact definitions). Denote by

$$F^c(u) = 1 - F(u)$$

the complement to F which is also called the *tail* of the distribution F. Then the quantity $\Delta^-(x)$ can readily be obtained analytically (Theorem 2):

$$\Delta^-(x) = \frac{F^c(x)}{\kappa + F^c(x)}.$$

In order to find the upper relative deviation Δ^+ (see (1.11)), we should use more sophisticated arguments. These arguments are based on the following characterization of a subexponential distribution: the d.f. F is subexponential if and only if some function $\beta_F(x)$ constructed below in (2.22) tends to 0 as $x \to \infty$ (Corollary 1 to Theorem 1). The function β_F can readily be constructed if we know F or even some of its general properties. What is actually interesting and new is the fact that the relative error Δ^+ can typically be estimated in terms of β_F:

$$\Delta^+(x) \leq C^+\beta_F(x), \tag{1.17}$$

where the constant C^+ can also be calculated. We write "typically" as in all considered case (Pareto, lognormal, Weibull, and others) the formula (1.17) is valid (see subsection 2.5). However, we cannot guarantee that this is true in a general case. But we state also a general result in Theorems 3 (for ruin probability) and 6 (for the general scheme and, in particular, for the accumulated claim size distribution).

The estimate (1.17) and its generalizations is a new mathematical result obtained in the course of this research project.

The function β_F can readily be obtained from the severity distribution (the distribution of claim sizes). Therefore, in order to estimate Δ^+, it is sufficient to find C^+. Analytically, we only prove that such a constant exists. The proof of the existence of C^+ actually shows us how to build a routine to find C^+ numerically.

To conclude this subsection, let us emphasize that we elaborated an approach allowing us to obtain both analytically and numerically two-sided bounds for the ruin probability, Ψ^- and Ψ^+, satisfying the properties (1.7) to (1.9) or, equivalently, to estimate the relative errors of the bounds, Δ^- and Δ^+, defined in (1.10) and (1.11). These quantities allow us to answer the questions (i) and (ii) posed at the end of subsection 1.3 about the accuracy of the approximation of Ψ_{SE}.

1.6 Truncation method

As we have already mentioned, the approximation Ψ_{SE} is not accurate, in general. Therefore, it is necessary to find alternative approximations for the probability of ruin. One of such approximations is proposed in this research. It can be readily calculated in the range of the initial capital that is of interest in actuarial practice. This approximation is not asymptotically correct, in general (at least, we cannot prove this property), but leads to appropriate numerical routines for bounding the ruin probability in the case of large claims. The idea of the approach is simple. We just note that the value of the probability of ruin, for a prescribed initial reserve x, does not depend on the values taken by the distribution $F(u)$ defined in (1.16) for $u > x$ and truncate the distribution F at level x. The truncated d.f. satisfies the Cramér condition and staightforward arguments (see subsection 2.7) imply the desired estimate, which is valid for all values $u \leq x$:

$$\frac{(1-q)F^c(x)}{q+(1-q)F^c(x)} \leq \Psi(u) \leq \frac{(1-q)F^c(x)}{q+(1-q)F^c(x)} + e^{-\varepsilon(x)u}, \tag{1.18}$$

where $\varepsilon(x)$ is the adjustment coefficient for the truncated d.f. F.

1.7 Numerical results and basic conclusions

Section 3 contains numerical results illustrating qualitative and quantitative properties of the estimates listed above. These results can be summarised by the following main conclusions.

(i) The lower bound Ψ^- defined in (1.12) is pretty close to the approximation Ψ_{SE}. But it can be far from the actual ruin probability Ψ.

(ii) In some cases, the upper bound Ψ^+ defined in (1.13) is close to the actual value Ψ but it is necessary to do an additional investigation when this is the case.

(iii) Conclusions (i) and (ii) mean that the approximation Ψ_{SE} is typically inaccurate and cannot be recommended for practical usage, in general.

(iv) The truncation approach gives satisfactory approximations for moderate values of the initial reserve x (typical for practice) but its implementation for large x (having a theoretical rather than practical interest) can be inaccurate.

(v) The truncation method can indicate when the approximation Ψ_{SE} or bounds Ψ^- and Ψ^+ can be used.

These conclusions are not too encouraging but they are realistic. The reason behind them can be explained as follows. The subexponential property of probability distributions is defined as a *tail property* (see subsection 2.2). This means that whether a d.f. F belongs to the class of subexponential distributions depends on the *limiting behavior* of the tail of this d.f. In particular, the behavior of the d.f. over *any* finite interval says nothing about its subexponentiality. But, in reality, we deal with finite values of random variables. In such a situation, the subexponentiality is a sort of abstraction which can or cannot fit the reality. And even if the subexponentiality hypothesis is reasonable, the approximation Ψ_{SE} can be poor for the values of the initial reserve which are of interest to practice.

1.8 Presentation

The results of the research were widely discussed at various scientific meetings. Here is the list of the most important presentations:

1. International Conference "Actuarial Science: Theory, Education, Implementation", October 1997, Moscow, Russia;

2. Seminar on Probability Theory, Moscow State University, December 1997, Moscow, Russia;

3. Third St.-Petersburg Workshop on Simulation, June 28 - July 3, 1998, St.-Petersburg, Russia;

4. International Seminar on Stability Problems for Stochastic Models, September 1998, Vologda, Russia;

5. International Conference on Probability Theory and Mathematical Statistics, August 1998, Vilnius, Lithuania;

6. Scientific seminar of the Dept. of Computer Science, University of Trier, May 1998, Trier, Germany;

7. Scientific Seminar of Mathematical Dept., Institute für Angewandte Mathematik und Statistik, University of Würzburg, May 1998, Würzburg, Germany;

8. The Dobrushin Mathematical Seminar, Institute for Information Transmission Problems, Russian Academy of Sciences, April 1998, Moscow, Russia;

9. Scientific Seminar of the Dept. of Mathematical Sciences, University of Aarhus, September 1998, Aarhus, Denmark.

The list of publications contain 4 papers [27, 26, 28, 29], 3 proceedings [22, 24, 25], and 3 preprints [17, 21, 23]. The material of preprints [17] and [21] was included into paper [27], and preprint [23] was updated for paper [28]. Papers [26, 29] are written in Russian. We provide the translation of crucial pieces of [29] and attach it to this report together with a copy of the manuscript of the paper. Paper [26] is a sort of survey and contains the results presented in [27, 28]; because of this, its translation is not necessary.

Important remark. Our preliminary results were presented to CKER in the form of two technical reports [18, 19]. As we continued working on the project until the last moment and as we tried to simplify our preceding results, the notation used in this report is slightly different than the one used in [18, 19]. This should not create any inconvenience since the final report is self-contained.

2 Mathematical background

2.1 Mathematical model

We start with some assumptions concerning the dynamics of the risk portfolio and the insurer's surplus, and introduce useful notations.

State these assumptions in the form of a risk model described as follows. Let $R(t)$ be the *surplus* of an insurer at time t and assume that its initial capital is x that is, $R(0) = x$. Let $c > 0$ be a *constant gross premium rate*, $\{Z_i\}_{i\geq 1}$ be *successive claim sizes*, $\{T_i\}_{i\geq 1}$ be *successive occurrence times*, $\{\theta_i\}_{i\geq 1}$ be *successive inter-occurrence times*: $\theta_1 = T_1$, $\theta_k = T_k - T_{k-1}$, $k \geq 1$, and N_t be *the number of claims until time t*. The dynamics of the insurer's surplus $R(t)$ is then described by the equation

$$R(t) = x + ct - \sum_{k\leq N_t} Z_k. \tag{2.1}$$

Since $\{T_i\}$ and $\{Z_i\}$ are usually random, $\{R(t)\}_{t\geq 0}$ is a random process. Such a model is called *the Sparre Andersen risk model* (see Grandell [12] and Kalashnikov [15]). Let $a_k = \mathbf{E}\theta^k$ and $b_k = \mathbf{E}Z^k$, $k \geq 1$, be the corresponding power moments of inter-occurrence times and claim sizes respectively. We assume also that *the relative safety loading*

$$\kappa = \frac{ca_1}{b_1} - 1$$

is *positive*, which guarantees the positive drift of the surplus which is a necessary condition for any successful insurance business.

One of the characteristics of our interest is the ruin probability $\Psi(x)$ defined in (1.2). The following representation of the probability of ruin as the right tail distribution is well-known (see Beekman [1] and Feller [11]) and it is crucial:

$$\Psi(x) = \mathbf{P}\left(\sum_{k\leq\nu} X_k > x\right), \quad x \geq 0, \tag{2.2}$$

where $\{X_i\}$ is a sequence of i.i.d.r.v.'s having a common d.f. $F(u) = \mathbf{P}(X_i \leq u)$ and where the r.v. ν is independent of $\{X_i\}$ and has the geometric distribution

$$\mathbf{P}(\nu = k) = q(1-q)^k, \quad k \geq 0. \tag{2.3}$$

If N_t is the Poisson process with the papameter λ, then both probability q and the d.f. F can easily be expressed in terms of initial data:

$$q = \frac{\kappa}{1+\kappa}, \tag{2.4}$$

$$F(u) = \mathbf{P}(X_1 \leq u) = \frac{1}{b_1}\int_0^u (1 - B(z))\,dz \tag{2.5}$$

(see [1, 12, 15]). Note that formula (2.5) is the same as formula (1.16). This model (called *classical*) is of special importance since the Poisson process N_t can often be justified in the case where occurrence times are formed as a superposition of comparatively rare individual claims.

If one considers a general S. Andersen model, then the main conclusion is the same, except that N_t does not have a Poisson distribution anymore, and the expressions of q and F differ from those given in (2.4) and (2.5) (this difference has already been discussed in details in our report [16]) and, actually, cannot be written explicitly.

If there exists $\varepsilon_C > 0$ such that

$$\mathbf{E}\exp(\varepsilon_C(Z_i - \theta_i)) = 1, \tag{2.6}$$

then the constant ε_C is called the *adjustment coefficient* or the *Cramér-Lundberg exponent*. The relation (2.6) is called the *Cramér condition*. It necessarily requires that a generic claim size Z has an exponential moment, at least. Such claims are called *small*.

Under the Cramér condition, the probability of ruin has the nice Cramér-Lundberg asyptotic approximation Ψ_{CL} (see (1.3)), where the Cramér-Lundberg k_{CL} is typically close to 1. It turns out (see [12, 15]) that real values of $\Psi(x)$ are pretty close to their asymptotic values $k_{CL}\exp(-\varepsilon_C x)$ even for moderate initial values of initial capital x. This is why the asymptotic approximation (1.3) can be used in actuarial calculations withour fear of making large errors.

In this research, we confine ourselves to the case of d.f.'s F called *subexponential* which are of special interest in insurance: severities arising from catastrophes such as tornados, earthquakes, floods, etc. can typically be described by subexponential distributions (*SE-distributions*), see [9]. Leaving precise definitions of SE-distributions for subsection 2.2, we note here that, for SE-distributions, one can derive an asymptotic formula which replaces, in a way, the Cramér-Lundberg approximation (see [9, 10, 15]):

$$\Psi(x) \sim \frac{1-q}{q} F^c(x). \tag{2.7}$$

This formula is a generalization of (1.4) and it is still simple and can easily be incorporated into numerical routines. But, as we have already mentioned, the accuracy of the approximation (2.7) can be poor for the values of initial capital which are of actuarial interest (see [20]).

Now, let us consider the accumulated claim distribution. Let C_t be the total i.i.d. claims accumulated during the accounting period t and defined in (1.1). The number N_t of occurrences is random, in general. For example, it has the Poisson distribution for the classical risk model. If the severity distribution is SE (the case of our interest), then the following asymptotic formula is valid

$$\mathbf{P}(C_t > x) \sim \mathbf{E}N_t\, B^c(x), \tag{2.8}$$

where B is the common d.f. of claim sizes (see [9]). Formula (2.8) is valid if there exists $\rho > 1$ such that $\mathbf{E}\rho^{N_t} < \infty$, which is a non-restrictive condition. Accuracy estimates of the relation (2.8) are also more than desirable.

We see that both the probability of ruin and the accumulated claims distribution can be represented in the *same form*

$$T(x) = \mathbf{P}\left(\sum_{k \leq \nu} X_k > x\right), \tag{2.9}$$

where $\{X_k\}$ is a sequence of i.i.d.r.v.'s having a common d.f. F. *We assume that F is an SE-distribution throughout the research*, and ν is an integer r.v. independent of $\{X_k\}$ and having the distribution

$$p_k = \mathbf{P}(\nu = k). \tag{2.10}$$

Actually, we regard $T(x)$ as the *basic mathematical model* to be studied.

When investigating the ruin probability, we assume that the distribution $\{p_k\}$ is geometric of the form (2.3). In general, we only assume that an arbitrarily distributed ν has a finite mean

$$\mu = \sum_{k=0}^{\infty} kp_k < \infty, \tag{2.11}$$

and satisfies the condition

$$\sum_{k=0}^{\infty} \rho^k p_k < \infty \tag{2.12}$$

for some $\rho > 1$. The condition (2.12) (which, in particular, implies (2.11)) requires that probabilities p_k decrease exponentially fast, at least, which is the case for many practically used distributions, for example, the Poisson one. If the r.v. ν is the number of clams occurred within a finite accounting period, then the condition (2.12) is always true. In this case, the asymptotic relation

$$T(x) \sim \mu F^c(x) \equiv T_{SE}(x) \tag{2.13}$$

holds (see [9]) which is a generalization of both (2.7) and (2.8).

2.2 Subexponential distributions

Let us start with the formal definition: F is called an SE-distribution, if

$$\lim_{u \to \infty} \frac{F_k^c(u)}{F^c(u)} = k \text{ for all } k \geq 2, \tag{2.14}$$

where F_k stands for the k-fold convolution of F, $F^c = 1 - F$, and $F_k^c = 1 - F_k$. This definition was proposed by Chistyakov [5] (see also [9]). It was proved in [5] that (2.14) holds for all k if and only if it holds for only $k = 2$. It was also proved in [5] that any exponential moment of the form

$$\int_0^\infty e^{au}\, dF(u), \quad a > 0,$$

is *infinite* as soon as $F \in SE$. This explains the name of the class.

The importance of the above definition for actuarial science can be explained by the fact that, in this case,

$$\mathbf{P}(X_1 + \cdots + X_k > u) \sim \mathbf{P}(\max(X_1, \ldots, X_k) > u) \tag{2.15}$$

which actually means that the sum of SE i.i.d.r.v.'s has asymptotically the same distribution as their maximum, or that the overwhelming contribution into the sum $X_1 + \cdots + X_k$ is made by only *one* summand. Referring to the ruin probability, this implies that the ruin event occurs mostly due to one large claim rather than the accumulation of small claims. And this phenomena can even be demonstrated by simulation experiments (see [33]).

The following *tail-equivalence property* of SE-distributions explains many phenomena occurring in the presence of SE-distributions and, in particular, some features of the two-sided bounds proposed in this research: *If $F \in SE$ and the d.f. G is such that $G^c(u) \sim aF^c(u)$ for a positive constant a, then $G \in SE$.* This explains why all subexponential effects can only occur for large x and are often not noticeable during finite intervals. It also explains why the Cramér–Lundberg theory is successfully applied even if it formally cannot. This also highlights why asymptotic formulas derived for the case of large claims can lead to large errors.

We are interested in exactly these situations which are very unpleasant in actuarial practice.

Sometimes, it is not easy to verify whether a d.f. F is subexponential or not using the formal definition. Because of this, we present necessary and sufficient conditions (characterizations) for F to belong to the class of SE-distributions. These characterizations were proved in our works [23, 28] and here we only formulate the corresponding conditions placing the emphasis on their potential applications and constructive consequences. These conditions are used for construction of the desired two-sided bounds Ψ^- and Ψ^+ (see (1.12) and (1.13). It is necessary to mention that the results were mostly stimulated by the profound work by E.S. Murphree [31].

The following quantities, defined for all $0 \le R \le x/2$, play a crucial role in the sequel:

$$I(x) \quad = \quad \frac{(F^c(x/2))^2}{F^c(x)}, \tag{2.16}$$

$$
\begin{aligned}
J(x,R) &= \frac{1}{F^c(x)}\int_R^{x/2} F^c(x-y)\,dF(y), && (2.17)\\
K(x,R) &= \frac{F^c(x-R)}{F^c(x)} - 1, && (2.18)\\
D(x,R) &= I(x) + J(x,R). && (2.19)
\end{aligned}
$$

Note that these quantities can readily be calculated (or, estimated) if we know the d.f. F. Investigating their behavior at $x \to \infty$, we can judge whether $F \in SE$ or not, and estimate the rate of convergence of the asymptotic approximations to the real tail distribution.

Let us introduce the following class of functions

$$
\mathcal{R} = \left\{ R(x) : \lim_{x\to\infty} R(x) = \infty \text{ and } 0 \le R(x) \le \frac{x}{2} \right\} \tag{2.20}
$$

Theorem 1. *$F \in SE$ if and only if there exists a function $R(x) \in \mathcal{R}$ such that*

$$
D(x, R(x)) \to 0, \quad K(x, R(x)) \to 0 \tag{2.21}
$$

as $x \to \infty$.

Theorem 1 is the principal result of this section. But it is unclear how to apply it and, in particular, how to find the desired function $R(x)$. The following corollary answers this question.

Before stating the corollary, note that if r is fixed, then $D(x,r) \to F^c(r)$ as $x \to \infty$. Evidently, $D(x,r)$ is a decreasing function and $K(x,r)$ is an increasing function of r, given x. Furthermore, the d.f. $F(x)$ is right-continuous (by definition). Therefore, $D(x,r)$ is right-continuous and $K(x,r)$ is left-continuous with respect to r.

Corollary 1. *Let*

$$
\beta_F(x) = \inf_{r \le x/2} \max\left(D(x,r),\, K(x,r)\right) \tag{2.22}
$$

and

$$
R^*(x) = \sup\left\{ r : r \le \frac{x}{2},\, D(x,r) \ge K(x,r) \right\}. \tag{2.23}
$$

Then

(i) $F \in SE$ if and only if $\beta_F(x) \to 0$;

(ii) any $F \in SE$ or $\beta_F(x) \to 0$ implies $R^*(x) \to \infty$.

The idea behind this corollary is clear. Let us imagine, for a moment, that we can solve the equation

$$
D(x,r) = K(x,r). \tag{2.24}
$$

Then its solution $R^*(x)$ belongs to the class $\mathcal{R}$,

$$
\beta_F(x) = D(x, R^*(x)) = K(x, R^*(x)),
$$

and, therefore,

$$\beta_F(x) \to 0$$

implying that $F \in SE$. As we will see, the quantity $\beta_F(x)$ actually estimates the proximity between the approximation (2.13) and the unknown tail distribution $T(x)$. This "naive" approach works in all examples below. But it is necessary to call the reader's attention to the following features which do not allow us to use the naive idea (without any corrections) in the general case. *First*, the equation (2.24) may have no solution, in general, as both functions D and K are not necessarily continuous. Therefore, we should replace equation (2.24) by something more appropriate and this is done in Corollary 1 (see (2.23)). *Secondly*, the quantity $R^*(x)$ is usually difficult to find from (2.23) since explicit analytical expressions of the functions $D(x,r)$ and $K(x,r)$ are not available, as a rule. We have to replace these explicit expressions by other functions, say, by majorants or asymptotic approximations of D and K. For instance, in some examples below we take $R^*(x)$ satisfying the asymptotic relation $D(x, R^*(x)) \sim K(x, R^*(x, R(x))$. In this case, we prove the following auxiliary result (which is a combination of Lemmas 3, 4, and 5 from [28]).

Lemma 1. *If $F \in SE$, then there exists a concave function $R(x) \in \mathcal{R}$ such that, for sufficiently large $x \geq x_D$,*

$$D(x, R(x)) \leq 2F^c(R(x)), \quad K(x, R(x)) \leq 2F^c(R(x)), \tag{2.25}$$

implying that

$$\beta_F(x) \leq 2F^c(R(x)), \quad x \geq x_D. \tag{2.26}$$

Usually we can write the asymptotic approximations or majorants of $D(x, R(x))$ and $K(x, R(x))$ (see examples below) and we definitely know $F^c(R(x))$. Therefore, relations (2.25) give us the possibility to define $R(x)$. Evidently, this definition is not unique, but the uniqueness is not necessary. This construction, applied to the examples below, gives the same result as the naive approach which ensures that it is close to optimal, at least. Furthermore, relations (2.25) give us the real opportunity to find $R(x)$, using, for example, majorants or asymptotic approximations of D and K.

Let us illustrate this by examples. We only display the main steps of the calculations. The details can be found in our paper [28, Section 4].

2.3 Examples

Let us consider three standard examples of SE-distributions which are widely used in actuarial practice and show how we can characterize them by the proposed criteria.

Example 1. Integrated Pareto tail distribution
Let us consider the classical risk model when the claim sizes have a *Pareto distribution*:

$$B(u) = \begin{cases} 0, & \text{if } u < \kappa, \\ 1 - (\kappa/u)^\alpha, & \text{if } u \geq \kappa, \end{cases}$$

where $\kappa > 0$ and $\alpha > 1$. Denote by

$$b_1 = \frac{\kappa\alpha}{\alpha - 1}$$

its mean value. Let

$$F(u) = \frac{1}{b_1} \int_0^u (1 - B(z))\, dz \tag{2.27}$$

be the corresponding *integrated Pareto tail distribution*. Then the probability of ruin can be expressed in the form (2.2) where X_i have the common integrated Pareto tail distribution F which is subexponetial (this fact is well-known, see [9], but it also follows from our results).

Let us estimate first the quantities I, J, and K involved in the characterization criteria. In this case,

$$I(x) = \frac{1}{\alpha} \left(\frac{4\kappa}{x}\right)^{\alpha-1}, \quad x > 2\kappa$$

and

$$J(x, R) \leq 2^{\alpha-1} F^c(R) = \frac{1}{\alpha} \left(\frac{2\kappa}{R}\right)^{\alpha-1} \equiv J^*(x, R), \quad \kappa \leq R \leq x/2.$$

Let

$$D^*(x, R) = I(x) + J^*(x, R).$$

If $\kappa \leq R \leq x/2$,

$$K(x, R) = \left(\left(1 - \frac{R}{x}\right)^{-\alpha+1} - 1\right) \leq \frac{(\alpha - 1)x^{\alpha-1}R}{(x - R)^\alpha} \equiv K^*(x, R).$$

Let us show how the naive idea works. Assume that $R(x) = o(x)$. Then

$$D^*(x, R(x)) \sim \frac{1}{\alpha} \left(\frac{2\kappa}{R(x)}\right)^{\alpha-1}$$

and

$$K^*(x, R(x)) \sim \frac{(\alpha - 1)R(x)}{x}.$$

Equating the right-hand sides in the last two relations, we arrive at

$$R^*(x) = c_R x^{1/\alpha}, \quad K^*(x, R^*(x)) \sim D^*(x, R^*(x)) \sim \frac{c_\beta}{x^{(\alpha-1)/\alpha}},$$

where

$$c_R = \left(\frac{(2\kappa)^{\alpha-1}}{\alpha(\alpha-1)}\right)^{1/\alpha}, \quad c_\beta = \frac{1}{\alpha}\left(2\kappa\alpha(\alpha-1)\right)^{(\alpha-1)/\alpha}.$$

Now, we know the limiting shape of $R^*(x)$ and, in accordance with this and omitting an unimportant constant c_R, we choose

$$R(x) = x^{1/\alpha}.$$

Using explicit formulas for D^*, we then find that

$$\max(D^*(x, R(x)), K^*(x, R(x)) \le \beta_F(x), \quad x \ge x_g,$$

where

$$\begin{aligned}
\beta_F(x) &= \frac{c_g}{x^{(\alpha-1)/\alpha}}, \\
c_g &= 2^\alpha \max\left(\frac{\kappa^{\alpha-1}}{\alpha}, \alpha - 1\right), \\
x_g &= \max(2^{\alpha/\alpha-1}, \kappa^\alpha).
\end{aligned}$$

Evidently, $\beta_F(x) \to 0$ as $x \to \infty$ and this gives an additional evidence that the integrated Pareto distribution belongs to the class of SE-distributions. For us, the most interesting thing is the exact value of $\beta_F(x)$ which will be later incorporated into the convergence rate estimates. Note that R is a concave function and $\beta_F(x)$ is asymptotically proportional to $F^c(R(x))$.

Example 2. Weibull distribution

Now, let us consider a *Weibull distribution* having the form

$$F(x) = 1 - \exp\left(-\lambda x^\beta\right),$$

where $\lambda > 0$, $0 < \beta < 1$, and $x \ge 0$. The condition $0 < \beta < 1$ implies that it is an SE-distribution.

Let us estimate the quantities I, J, and K in terms of which the characterization criteria are stated. Evidently,

$$I(x) = \exp\left(-\lambda x^\beta\left(2^{1-\beta} - 1\right)\right).$$

If $R(x) = o(x)$, then one can prove that

$$J(x, R(x)) \le J^*(x, R(x)) \sim \exp\left(-\lambda R^\beta(x)\right) = \frac{1}{\lambda^{1/\beta} x}$$

and

$$K(x, R) \le \exp\left(\frac{\lambda R}{x^{1-\beta}}\right) - 1 \equiv K^*(x, R).$$

Equating $D^*(x,R) = I(x) + J^*(x,R)$ and $K^*(x,R)$, we arrive at

$$R(x) = \left(\frac{1-\beta-\mu(x)}{\lambda}\ln\left(\lambda^{1/\beta}x\right)\right)^{1/\beta},$$

where

$$\mu(x) = \frac{\ln\ln\left(\lambda^{1/\beta}x\right)}{\beta\ln\left(\lambda^{1/\beta}x\right)},$$

from where we have

$$D(x, R(x)) \le 2F^c(R(x))$$

and

$$K(x, R(x)) \le F^c(R(x))$$

for sufficiently large x and, therefore,

$$\beta_F(x) = 2F^c(R(x)) = \frac{2\ln^{1/\beta}\left(\lambda^{1/\beta}x\right)}{(\lambda^{1/\beta}x)^{1-\beta}}.$$

This means that, in this case, Lemma 1 gives the precise estimate of $\beta_F(x)$.

Example 3. Lognormal distribution
Let us consider a *lognormal distribution* of the form

$$F(x) = \Phi(\ln x),$$

where Φ stands for a standard normal distribution. We restrict ourself to the standard case in order to avoid having to introduce additional notations. The general case is treated in [28]. Denote, for $x \ge 1$,

$$\begin{aligned} a(x) &= 1 - \frac{\ln 2}{\ln x}, \\ b(x) &= \frac{\ln^2 x}{1+\ln^2 x}. \end{aligned}$$

Obviously,

$$a(x) \to 1, \quad b(x) \to 1 \tag{2.28}$$

as $x \to \infty$.

In this case, we also start with estimating the quantities I, J, and K. One can get that

$$I(x) \le \frac{1}{\sqrt{2\pi a^2(x)b(x)}\ln(x)}\exp\left(-\ln^2\left(\frac{x}{2}\right)+\frac{\ln^2 x}{2}\right) \equiv I^*(x),$$

$$J(x,R) \le \frac{1}{a(x)b(x)} \int_R^{x/2} \exp\left(\frac{\ln^2 x - \ln^2(x-y)}{2}\right) dF(y) \equiv J^*(x,R),$$

and

$$K(x,R) \le \frac{\ln x}{\ln(x-R)} \exp\left(-\ln\left(1-\frac{R}{x}\right)\ln x\right) - 1 \equiv K^*(x,R).$$

Equating $D^*(x,R) = I^*(x) + J^*(x,R)$ and $K^*(x,R)$, we find that $R(x)$ has the form

$$\ln R(x) = \sqrt{2\ln x}\left(1 - \frac{1}{\sqrt{2\ln x}} - \frac{3\ln\ln x}{4\ln x} - \frac{1}{2\ln x}(1-\ln 2\sqrt{\pi})\right),$$

yielding

$$\beta_F(x) = \frac{R(x)\ln x}{x}.$$

Note that, in this case,

$$K(x,R(x)) \sim K^*(x,R(x)) \sim D^*(x,R(x)) \sim J^*(x,R(x)) \sim F^c(R(x)) \sim \beta_F(x)$$

which also shows that the estimate from Lemma 1 is precise.

2.4 Bounds for geometric sums

The material of this section is based on the results of our works [27, 23, 28]. Here, we consider the case where T is the tail distribution of a sum of i.i.d.r.v.'s $X_1, \ldots, X_\nu$, where the number of summands ν is also a r.v. having a geometric distribution with the parameter q:

$$p_k = \mathbf{P}(\nu = k) = q(1-q)^k, \quad k \ge 0,$$

see (2.9). Recall that this form of T arises when studying ruin probabilities for the classical and S.ndersen risk models. We build lower and upper bounds, $T^-(x)$ and $T^+(x)$, such that they embrace both the unknown function T and its known asymptotic approximation T_{SE} (see (2.13)) that is,

$$T^-(x) \le \quad T(x) \quad \le T^+(x), \tag{2.29}$$

$$T^-(x) \le \quad T_{SE}(x) \quad \le T^+(x), \tag{2.30}$$

and have proper asymptotic behavior:

$$T^-(x) \sim T^+(x) \sim T_{SE}(x). \tag{2.31}$$

Let us characterize these bounds by the following non-negative quantities

$$\Delta^-(x) = \frac{T_{SE} - T^-(x)}{T_{SE}(x)} \tag{2.32}$$

and

$$\Delta^+(x) = \frac{T^+(x) - T_{SE}}{T_{SE}(x)} \tag{2.33}$$

representing the *relative accuracy* of the asymptotic approximation $T_{SE}(x)$ that can be guaranteed. Because of the asymptotic equivalence property (2.31), both $\Delta^-(x)$ and $\Delta^+(x)$ tend to 0 as $x \to \infty$. When dealing with the lower bound, we do not use the assumption that $F \in SE$. Thus, this bound is universal and can be used for any d.f. F (not only subexponential). The following theorem discloses the lower bound and its accuracy.

Theorem 2. *For any d.f. F and any $x \geq 0$,*

$$T(x) \geq T^-(x) = \frac{(1-q)F^c(x)}{q + (1-q)F^c(x)}. \tag{2.34}$$

and

$$\Delta^-(x) = \frac{T_{SE}(x) - T^-(x)}{T_{SE}(x)} = \frac{(1-q)F^c(x)}{q + (1-q)F^c(x)}. \tag{2.35}$$

Note that $T^-(0) = T(0) = 1-q$ and, evidently, $T^-(x) \sim T_{SE}(x)$. If $F \in SE$, then $T^-(x) \sim T(x)$ and therefore $T^-(x)$ is close to $T(x)$ for also large x. If x is moderate, then the accuracy of $T^-(x)$ depends on the shape of F. Note also that an analog of the bound (2.34) was found in [6] by De Vylder and Goovaerts for the classical risk model.

The bound (2.34) can be calculated readily and the value (2.35) of the relative error incidentally coincides with the value (2.34) of the bound. Therefore, the lower bound is pretty close to (2.13) and this explains why $T_{SE}(x)$ is often too "optimistic" (see [15, 20]).

In order to derive an upper bound, we should use more sophisticated arguments. The following result is closely related to Corollary 1 and Lemma 1.

Lemma 2. *If $F \in SE$, then there exists a function $R(x) \in \mathcal{R}$ satisfying (2.21), a positive monotonically decreasing function g and a constant $x_g \geq 0$ such that*

$$g(x) \to 0 \text{ as } x \to \infty, \tag{2.36}$$

$$g(x) \geq \max\left(D(x, R(x)), K(x, R(x))\right), \quad x \geq x_g, \tag{2.37}$$

$$\frac{g(x - R(x))}{g(x)} \to 1 \text{ as } x \to \infty. \tag{2.38}$$

Actually, the value $\beta_F(x)$ defined in (2.22) satisfies the properties (2.36) and (2.37) (with $R(x) = R^*(x)$ defined by (2.23)) but it is unclear whether it satisfies (2.38) which is important for the sequel. But if we take $g(x) = 2F^c(R(x))$ as in Lemma 1, then all conditions (2.36) to (2.38) are satisfied. Let us mention that the function $g(x)$

thus defined depends on only the d.f. F and *does not depend* on q or, more generally, on the distribution $\{p_k\}$ of the number of summands (when this distribution is not geometric). This fact will be exploited in subsection 2.6.

It turns out that any function $g(x)$ satisfying (2.36) to (2.38) gives an upper estimate of the relative accuracy

$$\Delta(x) \leq \Delta^+(x) = C^+ g(x) \tag{2.39}$$

as $x \geq x^*$, where C^+ and x^* are constants to be defined. We shall return to their definition when stating the result accurately. For now, it is sufficient to understand that, knowing $g(x)$, we have an upper bound of form (2.39) of the desired relative accuracy and the constants x^* and C^+ which determine the bound. Therefore, the crucial step is finding the appropriate function $g(x)$. But we have already discussed this problem in subsection 2.2. Particularly, we showed how to find appropriate bounds which are proportional to $F^c(R(x))$ and we do know that the function $g(x) = 2F^c(R(x))$ fits our purpose. In all examples considered above we take $g(x) = \beta_F(x)$. That is why we devoted so much effort to the characterization problem.

Now, let us return to the bound (2.39) and state several assertions highlighting it. Denote

$$f_1(x) = \frac{(1-q)g(x-R(x))}{g(x)} (K(x,R(x))+1), \tag{2.40}$$

$$f_2(x) = \frac{2(1-q)g(R(x))}{g(x)} D(x,R(x)), \tag{2.41}$$

$$f_3(x) = \frac{1}{g(x)} \left(2(1-q)D(x,R(x)) + K(x,R(x))\right). \tag{2.42}$$

Lemma 3. *If $F \in SE$ and functions R and g satisfy Lemma 2, then, for any $1-q < \delta^+ < 1$, there exist constants $x^* \geq x_g$ and $\varphi \leq 3-q$ such that*

$$f_1(x) + f_2(x) \leq \delta^+, \tag{2.43}$$

$$f_3(x) \leq \varphi, \tag{2.44}$$

for any $x \geq x^*$.

One can see that the quantity δ^+ affecting the accuracy can be taken rather freely within the interval $(1-q, 1)$. This gives us an additional degree of freedom to improve the bound. As our experience shows, the optimal choice of δ^+ is virtually impossible to obtain analytically. But we can actually find it numerically with the help of a computer. In all numerical results given below we did so. Actually, Lemma 3 defines x^* depending on δ^+: the larger δ^+ is, the smaller x^* is. This gives the left abscissa of applicability of the upper bound.

Now, in order to define C^+, let us introduce the following quantities

$$\Delta^*(x) = \max\left\{0, \frac{T(x) - T_{SE}(x)}{T_{SE}(x)}\right\}, \tag{2.45}$$

$$C^*(x) = \frac{\Delta^*(x)}{g(x)}, \tag{2.46}$$

$$C^*[a,b] = \max_{a \le x \le b} C^*(x). \tag{2.47}$$

The next theorem contains the desired upper bound of $\Delta(x)$.

Theorem 3. *If $F \in SE$, functions R, g satisfy the conditions of Lemma 2, and constant x^* is taken from Lemma 3. Then (2.39) holds with*

$$C^+ = \max\left(\frac{\varphi}{1-\delta^+}, \varphi + \delta^+ C^*[R(x^*), x^*]\right). \tag{2.48}$$

Let us call attention to the following important and somewhat misleading feature of the bound (2.39). Being defined by (2.48), the constant C^+ depends on the unknown function $T(x)$ (see (2.45) through (2.47)) which should be estimated. Fortunately, this is not a closed circle: C^+ depends on $C^*[R(x^*), x^*]$ which, in turn, depends on $T(x)$, $x \in [R(x^*), x^*]$, whereas (2.39) represents a bound of $T(x)$ for $x \geq x^*$. Thus, the final estimate (2.39) of $\Delta(x)$ *over the infinite interval* $x \in [x^*, \infty)$ depends on the values taken by $\Delta(x)$ *within the finite interval* $x \in [R(x^*), x^*]$. Such a situation is typical for various applied problems. Let us refer to the continuity analysis of general Markov chains or queueing models where the infinite horizon continuity estimate incorporates some finite horizon continuity bound (see [14]). This has the following consequence: in order to use bound (2.39) we should be able to estimate $T(x)$ for $x \leq x^*$ (or, more precisely, for $R(x^*) \leq x \leq x^*$). Crude analytical bounds for $T(x)$, $x \leq x^*$, can be stated easily and they can be found in our paper [27]. For example,

$$T(x) \leq (1-q)(qF^c(x) + (1-q)).$$

But numerical calculations made it evident that a crude estimate of $T(x)$, $x \leq x^*$, yield a crude estimate of C^+ (see corresponding examples in Section 3). Therefore, the problem is how to obtain *tight bounds* of $T(x)$ over the finite interval. During this research, we proposed one of such methods which is discussed in Section 2.7. Now, we list other possible alternatives to do this. First, let us mention the recursive algorithm refined by Dufresne and Gerber in [7] which gives lower and upper bounds of $T(x)$ for any finite x (at least, theoretically). In practice, this algorithm is fast and accurate for moderate values of x and comparatively large values of q ($q \geq 0.1$). For small q or large x, this algorithm becomes time-consuming and leads to large errors (see [15, 16, 20]). Second, we can employ the bounds proposed in [15, 20] which are not asymptotically correct but work reasonably well over finite intervals.

To conclude this subsection, let us list the basic steps to be done to find the upper bound Δ^+.

(1) Given the d.f. F, calculate quantities D and K defined in (2.19) and (2.18).

(2) Using Lemma 1, find the function $R(x)$.

(3) Define $g(x)$ satisfying the conditions of Lemma 2.

(4) Playing with δ^+ and φ, find x^* as small as possible (see Lemma 3).

(5) Estimate $C^*[R(x^*), x^*]$ by any available method (as accurate as possible).

(6) Define C^+ by formula (2.48).

(7) The desired upper bound Δ^+ has the form (2.39).

2.5 Examples

Let us return to the examples listed in subsection 2.3. In all cases, the functions $g(x) = \beta_F(x)$ and $R(x)$ which were then obtained satisfy the conditions of Lemma 2. The constant x^* is defined by Lemma 3, where δ^+ is chosen numerically to minimize x^*. The constant C^+ is defined by Theorem 3 and we have already discussed some problems associated with its calculation in subsection 2.4. The results of all calculations are presented below in Section 3.

2.6 Bounds for the general case

The material of this section originated from the work [29] where the corresponding proofs and auxiliary constructions can be found. We do not require anymore that the distribution $\{p_k\}$ be geometrical but only require that it satisfies the condition (2.12).

We start with lower bounds which are a little more complicated here than in the geometric case but still simple and valid for any d.f. F, not only for SE-distributions.

First, let us assume that the second moment is finite

$$\mu_2 = \mathbf{E}\nu^2 = \sum_{k=0}^{\infty} k^2 p_k < \infty. \tag{2.49}$$

Theorem 4. *If $\mu_2 < \infty$, then*

$$\Delta^-(x) \leq \frac{\mu_2}{\mu} F^c(x). \tag{2.50}$$

The requirement $\mu_2 < \infty$ imposes almost no restriction in practical cases. Nevertheless, from a mathematical point of view, it would be desirable to find the weakest possible conditions. It turns out that a finite mean, $\mu < \infty$, is sufficient for the result. In order to prove this, we need the following well-known criterion (see [15, 30]).

Proposition 1. *The mean is finite, $\mu < \infty$, if and only if there exists a non-negative increasing convex function $G(x)$ such that*

$$G(t)/t \uparrow \infty, \quad G(t)/t^2 \downarrow 0, \quad t \to \infty. \tag{2.51}$$

and $\mathbf{E}G(\nu) < \infty$.

The shape of the function G used in Proposition 1 depends, in general, on the distribution $\{p_k\}$. If, for example, $\mathbf{E}\nu^\alpha < \infty$ for some $\alpha > 1$, then G can be taken in the form $G(x) = x^\alpha$.

Theorem 5. *Let the distribution $\{p_k\}$ be such that $\mu_G \equiv \mathbf{E}G(\nu) < \infty$ for some function G satisfying (2.51). Then*

$$\Delta^-(x) \le \frac{2\mu_G}{\mu F^c(x) G(1/F^c(x))} \to 0, \quad x \to \infty. \tag{2.52}$$

In particular, if $\mu_\alpha = \mathbf{E}\nu^\alpha < \infty$ for some $1 < \alpha < 2$, then

$$\Delta^-(x) \le \frac{2\mu_\alpha}{\mu} \left(F^c(x)\right)^{\alpha-1}.$$

Theorems 4 and 5 give explicit expressions for asymptotically correct lower bounds. Of course, the statement of Theorem 4 is more attractive and it actually covers most cases arising in practice.

The upper bound, like in the geometric case, requires more sophisticated arguments. Luckily, most of the arguments have already been clarified in subsections 2.2 and 2.4 and here we pay attention to the existing difference between the two cases.

Our basic result consists of the following. We prove that, under condition (2.12), there exist constants x^* and C^+ such that

$$\Delta(x) \le \Delta^+(x) \le C^+ g(x), \quad x \ge x^*,$$

where $g(x)$ is defined in Lemma 2. As we have already mentioned, $g(x)$ does not depend on the distribution $\{p_k\}$. Note that the above inequality is just the same as (2.39) and the hidden difference lays in definitions of both C^+ and x^*. Let us disclose this.

Similarly to (2.45) through (2.47), let us define functions

$$\begin{aligned}
\Delta_k(x) &= \left(\frac{F_k^c(x)}{F^c(x)} - k\right)_+ \le \left(\frac{1}{F^c(x)} - k\right)_+ \le \frac{1}{F^c(x)}, \quad k \ge 1, \\
C_k(x) &= \frac{\Delta_k(x)}{g(x)} \\
C_k[a,b] &= \sup_{a \le x \le b} C_k(x).
\end{aligned}$$

It is important that *all* functions $C_k[0, x^*]$ have a *common* bound:

$$C_k[R(x^*), x^*] \le \overline{C}(x^*) \le \frac{1}{g(x^*)F^c(x^*)}. \tag{2.53}$$

The inequality (2.53) can be crude and, in calculations, it is better to seek more accurate bounds but the existence of the *uniform bound* is crucial for the sequel. Let x_g be defined by (2.37) and let

$$\begin{aligned}
f_1(x) &= \frac{g(x - R(x))}{g(x)} (K(x, R(x)) + 1), \\
f_2(x) &= \frac{g(R(x))}{g(x)} D(x, R(x)).
\end{aligned}$$

Evidently,

$$\begin{aligned}
f_1(x) &\to 1, \quad x \to \infty, \\
f_2(x) &\to 0, \quad x \to \infty.
\end{aligned}$$

Let $x^* \ge x_g$. Then

$$\begin{aligned}
\sup_{x \ge x^*} (f_1(x) + f_2(x)) &= \rho(x^*), \\
\sup_{x \ge x^*} \frac{D(x, R(x)) + K(x, R(x))}{g(x)} &= \delta(x^*) \le 2,
\end{aligned}$$

where functions both $\rho(x^*)$ and $\delta(x^*)$ are monotonically decreasing and

$$\rho(x^*) \to 1, \quad x^* \to \infty. \tag{2.54}$$

Lemma 4. *If $x^* \ge x_g$, then, for any $k \ge 1$,*

$$\sup_{y \ge x^*} C_k[x^*, y] \le \frac{\rho(x^*)(\rho^{k-1}(x^*) - 1)}{\rho(x^*) - 1} \left(\frac{\delta(x^*)}{\rho(x^*) - 1} + \overline{C}(x^*)\right) - \frac{\delta(x^*)(k-1)}{\rho(x^*) - 1},$$

where $\overline{C}(x^*)$ satisfies the inequality

$$\overline{C}(x^*) \le \frac{1}{g(x^*)F^c(x^*)}.$$

This lemma yields the following result.

Theorem 6. *Assume that the distribution $\{p_k\}$ satisfies (2.12) and functions g and R are chosen as in Lemma 2. Then the upper bound (2.39) holds with*

$$C^+ = \frac{1}{\mu}\sum_{k=1}^{\infty} p_k \left(\rho(x^*)(\rho^{k-1}(x^*)-1)\left(\frac{\delta(x^*)}{(\rho(x^*)-1)^2} + \frac{\overline{C}(x^*)}{\rho(x^*)-1}\right) - \frac{\delta(x^*)(k-1)}{\rho(x^*)-1}\right).$$

The constant C^+ thus defined is finite if $x^* \geq \max(x_g, x_\rho)$, where

$$x_\rho = \min\left\{t : \sum_{k=1}^{\infty} p_k \rho^k(t) < \infty\right\},$$

the convergence of this series is guaranteed by the conditions (2.12) and (2.54).

We limit ourselves to these theoretical results. Their applications can be stated quite similarly to the geometric case; see the last paragraph of subsection 2.4.

2.7 Truncation approach

The Cramér case

Hereafter, we deal with only $T(x)$ which is the tail distribution of a geometric sum that is, $\{p_k\}$ has the form (2.3). As we have seen, the calculation of the lower bound is fairly simple and poses no problem, and the most difficult part of calculation of the upper bound $\Delta^+(x)$ is that of $C^+[R(x^*), x^*]$ for which we have to obtain tight bounds of $T(x)$ over finite intervals.

Let us repeat that one can use several options to get such bounds. First, numerical calculations provided by the routine described in [7] (and we actually used that routine). Second, one can use the bounds obtained during our previous research (see [16, 15, 20]). They are not asymptotically correct but have reasonable accuracy for moderate values of x. The difference between these two types of bounds is the following. The recursive algorithm taken from [7] has the property to fail as $q \to 0$ or $x \to \infty$. Unlike this, the bounds proposed in [16, 15, 20] can be readily calculated for any $q > 0$ and $x > 0$. Furthermore, their accuracy increases as q tends to 0. But the corresponding bounds are not asymptotically correct and therefore, their relative accuracy falls as $x \to \infty$.

In [17, 27], we proposed another approach enabling us to approximate $T(x)$ for moderate x.

Remark. As we learned when preparing this report, the same approach was developed by J. Cai and J. Garrido who used our results for proving their constructions (see [3, 4]). They also did some calculations which completely agree with our results and confirm the good accuracy of the method proposed.

Note that in our works [17, 27] we obtained also some additional results (concerning the limiting behavior of the hazard rate of the reliability of regenerative models and the behavior of ruin probabilities in the case where the Cramér condition is violated) but now we only intend to discuss computational aspects of the approach.

The method itself is originated from the change of probability measures technique and it can be explained as follows.

Let us start with the case where F satisfies the Cramér condition (see Grandell [12] and Kalashnikov [15]): there exists $\varepsilon_C > 0$ such that

$$(1-q) \int_0^\infty e^{\varepsilon_C u} \, dF(u) = 1. \tag{2.55}$$

Note that (2.55) is equivalent to the condition (2.6). Define the distribution G by the relation

$$G(dx) = (1-q) e^{\varepsilon_C x} F(dx) \tag{2.56}$$

and introduce the *renewal process*

$$S_n = X_1 + \cdots + X_n, \quad n \geq 1, \tag{2.57}$$

where $\{X_i\}$ are i.i.d.r.v.'s having the common d.f. G. Let

$$N(x) = \min\{n : X_1 + \cdots + X_n > x\} \tag{2.58}$$

and

$$\eta(x) = S_{N(x)} - x$$

be the *excess of the renewal process* $\{S_n\}$ over level x.

In our works [17, 27] we obtained the following exact representation for the tail distribution $T(x)$.

Theorem 7. *If the Cramér condition (2.55) holds, then*

$$T(x) = e^{-\varepsilon_C x} \, \mathbf{E}_G e^{-\varepsilon_C \eta(x)}, \tag{2.59}$$

where $\mathbf{E}_G$ *indicates that the inter-renewal times have the common d.f.* G.

By its form, this representation is similar to the famous martingale representation by Gerber (see [12]). But (7) has some advantages in that the term $\mathbf{E}e^{-\varepsilon_C \eta(x)}$ is more tractable and there is no problem with its convergence. In all cases it does not exceed 1 thus yielding the famous Lundberg inequality (see [12, 15]). In works [17, 27] we showed that this representation implies the famous Cramér-Lundberg approximation and also can be employed in the cases where the Cramér condition is violated.

The subexponential case

Now, we are going to show how the representation (2.59) can be used for calculation of $T(x)$ in the case where F is an SE-distribution that is, the Cramér condition does not hold. In [17, 27] we indicated that this can be done by the truncation arguments.

Note that $T(x)$, for a fixed x, does not depend on values taken by $F(u)$ at $u \geq x$. This suggests two natural truncations. In both of them we assume that x is fixed.

(i) Let

$$F'(u) = \begin{cases} F(u), & \text{if } u < x, \\ 1, & \text{if } u \geq x. \end{cases}$$

Then F' satisfies the Cramer condition that is, there exists a positive constant $\varepsilon_1 = \varepsilon_1(x)$ such that

$$(1-q) \int_0^x e^{\varepsilon_1 u}\, dF'(u) = 1$$

which is equivalent to

$$(1-q)\left(\int_0^x e^{\varepsilon_1 u}\, dF(u) + e^{\varepsilon_1 x} F^c(x)\right) = 1. \tag{2.60}$$

Let

$$G'(u) = (1-q) \int_0^u e^{\varepsilon_1 z}\, dF'(z), \quad u \leq x,$$

(evidently, $G'(x) = 1$). Theorem 7 follows that

$$T(u) = e^{-\varepsilon_1 u}\, \mathrm{E}_{G'} e^{-\varepsilon_1 \eta(u)}, \quad u \leq x. \tag{2.61}$$

(ii) The second way is more sophisticated and more attractive. Let

$$F''(u) = \begin{cases} F(u)/F(x), & \text{if } u < x, \\ 1, & \text{if } u \geq x. \end{cases}$$

The k-fold convolution $F_k''(u)$ has the form

$$F_k''(u) = F_k(u)/F^k(x), \quad u \leq x.$$

Let us use this and rewrite $T(u)$, $u \leq x$, designating $q' = 1 - (1-q)F(x)$:

$$T(u) = \frac{(1-q)F^c(x)}{q + (1-q)F^c(x)} + \frac{q}{q'}\, T'(u),$$

where

$$T'(u) = q' \sum_{k=0}^{\infty} (1-q')^k (1 - F_k''(u))$$

The term T' represents a geometric convolution and, by Theorem 7,

$$T'(u) = e^{-\varepsilon_2 u}\, \mathbf{E}_{G''} e^{-\varepsilon_2 \eta(u)}, \quad u \le x,$$

where $\varepsilon_2 = \varepsilon_2(x)$ is the unique solution of the Cramér equation

$$(1-q') \int_0^x e^{\varepsilon_2 u}\, dF''(u) = 1$$

which is equivalent to

$$(1-q) \int_0^x e^{\varepsilon_2 u}\, dF(u) = 1 \tag{2.62}$$

and

$$G''(u) = (1-q) \int_0^u e^{\varepsilon_2 z}\, dF(z).$$

This results in

$$T(u) = \frac{(1-q)F^c(x)}{q + (1-q)F^c(x)} + e^{-\varepsilon_2 u}\, \mathbf{E}_{G''} e^{-\varepsilon_2 \eta(u)}, \quad u \le x. \tag{2.63}$$

The first summand in (2.63) represents the lower bound $T^-(x)$ obtained in Theorem 2 and it has the correct asymptotic behavior. Therefore, the second summand tends to 0 faster than the first one.

It is not easy, however, to use these explicit expressions as we have to calculate ε_i $(i = 1, 2)$ and $\mathbf{E}_{G'} e^{-\varepsilon_2 \eta(u)}$ or $\mathbf{E}_{G''} e^{-\varepsilon_2 \eta(u)}$.

As for the constants ε_i, they can be found from the corresponding Cramér's equations but we should remember that they depend on x and hence, these equations should be solved when x is fixed. The solution can be done numerically by any appropriate recursive method but special attention should be paid to the accuracy control as, for SE-distributions, $\varepsilon_i(x) \to 0$ as $x \to \infty$.

The terms $\mathbf{E}_{G'} e^{-\varepsilon_2 \eta(u)}$ or $\mathbf{E}_{G''} e^{-\varepsilon_2 \eta(u)}$ are much more difficult to estimate. We need their upper bounds and, evidently, we can take 1 as such bounds (actually, we did so). But this can lead to the loss of accuracy as it is unclear whether the remainder exponential terms $e^{\varepsilon_i(x)x}$ still tend to 0 faster than the first summand in (2.63) or not. We cannot answer this question and J. Cai and J. Garrido in [3, 4] do not answer it either. But numerical calculations show that such estimates work reasonably good for at least those values of x which are of interest to us in order to support calculations of the asymptotically correct bounds.

So, we propose to use the following two routines to find upper bounds $T^+(x)$.

(i) Solve (numerically) the Cramér equation (2.60) and take

$$T^+(x) = \exp(-\varepsilon_1(x)x). \tag{2.64}$$

(ii) Solve (numerically) the Cramér equation (2.62) and take

$$T^{+}(x)=\frac{(1-q)F^{c}(x)}{q+(1-q)F^{c}(x)}+\exp(-\varepsilon_{2}(x)x). \tag{2.65}$$

Numerical calculations show that both methods give fairly close results.

It is necessary to mention that equations (2.60) and (2.62) can also be solved in the case where the d.f. F is unknown but sample values of X_i are available. Then we can use the Monte-Carlo method to estimate ε_1 or ε_2. Some results in this direction are presented in the following section.

3 Numerical examples

The following approximations of the unknown function T are considered in this section:

$T^{-}_{DG}(x)$ – the lower bound obtained by the numerical routine from [7];

$T^{+}_{DG}(x)$ – the upper bound obtained by the numerical routine from [7];

$T^{+}_{GS}(x)$ – the upper bound obtained in [15, Theorem 4.3.2];

$T^{+}_{C1}(x)$ – the upper bound (2.64);

$T^{+}_{C2}(x)$ – the upper bound (2.65);

$T_{EV}(x)$ – the approximation taken from (2.7);

$T^{-}(x)$ – the lower bound (2.34).

The reader can have a vivid impression about their accuracy from the numerical results we present.

3.1 Pareto-like distribution

First, we consider the Pareto-like distribution

$$F(u)=1-(1+\beta u)^{-\alpha},\quad \alpha>0,\beta>0.$$

Tables 1 to 6 contain numerical results that illustrate the real behavior of $T(x)$ and its various bounds. In all tables the mean value of the d.f. F is taken as 1. Tables 1 to 3 refer to the case $\alpha=3$ and $\beta=0.5$ and Tables 4 to 6 deal with the case $\alpha=5$ and $\beta=0.25$. The latter d.f. has a lighter tail than the former one. Empty cells in these tables (referring to only T^{-}_{DG} and T^{+}_{DG}) indicate that the method proposed in

x	$T_{EV}(x)$	$T^-_{DG}(x)$	$T^+_{DG}(x)$	$T^-(x)$	$T^+_{C1}(x)$	$T^+_{C2}(x)$
1	$3.29 \cdot 10^{-2}$	$3.32 \cdot 10^{-2}$	$3.32 \cdot 10^{-2}$	$3.19 \cdot 10^{-2}$	$4.46 \cdot 10^{-2}$	$4.08 \cdot 10^{-2}$
3	$7.11 \cdot 10^{-3}$	$7.78 \cdot 10^{-3}$	$7.79 \cdot 10^{-3}$	$7.06 \cdot 10^{-3}$	$1.32 \cdot 10^{-2}$	$1.18 \cdot 10^{-2}$
5	$2.59 \cdot 10^{-3}$	$2.86 \cdot 10^{-3}$	$2.87 \cdot 10^{-3}$	$2.58 \cdot 10^{-3}$	$5.96 \cdot 10^{-3}$	$5.64 \cdot 10^{-3}$
7	$1.22 \cdot 10^{-3}$	$1.34 \cdot 10^{-3}$	$1.34 \cdot 10^{-3}$	$1.22 \cdot 10^{-3}$	$3.15 \cdot 10^{-3}$	$2.91 \cdot 10^{-3}$
9	$6.68 \cdot 10^{-4}$	$7.24 \cdot 10^{-4}$	$7.25 \cdot 10^{-4}$	$6.67 \cdot 10^{-4}$	$1.79 \cdot 10^{-3}$	$1.63 \cdot 10^{-3}$
10^2	$8.38 \cdot 10^{-7}$	$8.43 \cdot 10^{-7}$	$8.44 \cdot 10^{-7}$	$8.38 \cdot 10^{-7}$	$1.49 \cdot 10^{-6}$	$1.38 \cdot 10^{-6}$
10^3	$8.84 \cdot 10^{-10}$	$8.84 \cdot 10^{-10}$	$8.85 \cdot 10^{-10}$	$8.84 \cdot 10^{-10}$	$1.39 \cdot 10^{-9}$	$1.19 \cdot 10^{-9}$
10^4	$8.88 \cdot 10^{-13}$	$8.88 \cdot 10^{-13}$	$8.90 \cdot 10^{-13}$	$8.88 \cdot 10^{-13}$	$1.30 \cdot 10^{-12}$	$1.13 \cdot 10^{-12}$
10^5	$8.89 \cdot 10^{-16}$	–	–	$8.89 \cdot 10^{-16}$	$1.26 \cdot 10^{-15}$	$1.08 \cdot 10^{-15}$
10^7	$8.89 \cdot 10^{-22}$	–	–	$8.89 \cdot 10^{-22}$	$1.43 \cdot 10^{-21}$	$1.04 \cdot 10^{-21}$

Table 1: Pareto-like case: $q = 0.9$, $\alpha = 3$, $\beta = 0.5$

x	$T_{EV}(x)$	$T^-_{DG}(x)$	$T^+_{DG}(x)$	$T^-(x)$	$T^+_{C1}(x)$	$T^+_{C2}(x)$
1	2.67	$7.91 \cdot 10^{-1}$	$7.91 \cdot 10^{-1}$	$7.27 \cdot 10^{-1}$	$8.32 \cdot 10^{-1}$	1.06
3	$5.76 \cdot 10^{-1}$	$6.36 \cdot 10^{-1}$	$6.38 \cdot 10^{-1}$	$3.66 \cdot 10^{-1}$	$7.01 \cdot 10^{-1}$	$8.69 \cdot 10^{-1}$
5	$2.10 \cdot 10^{-1}$	$5.19 \cdot 10^{-1}$	$5.22 \cdot 10^{-1}$	$1.74 \cdot 10^{-1}$	$5.88 \cdot 10^{-1}$	$6.58 \cdot 10^{-1}$
7	$9.88 \cdot 10^{-2}$	$4.26 \cdot 10^{-1}$	$4.30 \cdot 10^{-1}$	$8.99 \cdot 10^{-2}$	$4.91 \cdot 10^{-1}$	$5.18 \cdot 10^{-1}$
9	$5.41 \cdot 10^{-2}$	$3.50 \cdot 10^{-1}$	$3.56 \cdot 10^{-1}$	$5.13 \cdot 10^{-2}$	$4.12 \cdot 10^{-1}$	$4.20 \cdot 10^{-1}$
10^2	$6.79 \cdot 10^{-5}$	$1.89 \cdot 10^{-4}$	$3.48 \cdot 10^{-4}$	$6.78 \cdot 10^{-5}$	$9.25 \cdot 10^{-4}$	$7.70 \cdot 10^{-4}$
10^3	$7.16 \cdot 10^{-8}$	$7.32 \cdot 10^{-8}$	$8.24 \cdot 10^{-8}$	$7.16 \cdot 10^{-8}$	$1.36 \cdot 10^{-7}$	$1.16 \cdot 10^{-7}$
10^4	$7.20 \cdot 10^{-11}$	$7.20 \cdot 10^{-11}$	$8.07 \cdot 10^{-11}$	$7.20 \cdot 10^{-11}$	$1.17 \cdot 10^{-10}$	$9.80 \cdot 10^{-11}$
10^5	$7.20 \cdot 10^{-14}$	$7.20 \cdot 10^{-14}$	$8.08 \cdot 10^{-14}$	$7.20 \cdot 10^{-14}$	$9.36 \cdot 10^{-14}$	$8.93 \cdot 10^{-14}$
10^7	$7.20 \cdot 10^{-20}$	–	–	$7.20 \cdot 10^{-20}$	$8.82 \cdot 10^{-20}$	$8.55 \cdot 10^{-20}$

Table 2: Pareto-like case: $q = 0.1$, $\alpha = 3$, $\beta = 0.5$

[7] fails to work in these cases. In all other cases, the bounds T^-_{DG} and T^+_{DG} embrace the real value $T(x)$. Note that in *all cases* $T_{EV}(x) \leq T^-_{DG}$ which means that $T_{EV}(x)$ should be regarded as a too optimistic approximation. We did the calculations for different values of x (even for those which are out of actuarial interest) to illustrate the heavy-tailed effects which can actually occur for large x.

Tables 1 and 4 show that, for large q ($q = 0.9$ which corresponds to the relative safety loading equal to 9), the asymptotic approximation T_{EV} works well even for small values of x. One can see also that the two proposed upper bounds T^+_{C1} and T^+_{C2} are close to each other and they are stable even for large x. The values of these upper bounds are larger than the real value of $T(x)$ by approximately 1.5 times. This is due to the substitution of the exact formulas (2.61) and (2.63) by their upper bounds

x	$T_{EV}(x)$	$T^-_{DG}(x)$	$T^+_{DG}(x)$	$T^-(x)$	$T^+_{C1}(x)$	$T^+_{C2}(x)$
1	29.3	$9.77 \cdot 10^{-1}$	$9.77 \cdot 10^{-1}$	$9.67 \cdot 10^{-1}$	$9.85 \cdot 10^{-1}$	1.38
3	6.34	$9.56 \cdot 10^{-1}$	$9.56 \cdot 10^{-1}$	$8.64 \cdot 10^{-1}$	$9.65 \cdot 10^{-1}$	1.59
5	2.31	$9.36 \cdot 10^{-1}$	$9.36 \cdot 10^{-1}$	$6.98 \cdot 10^{-1}$	$9.48 \cdot 10^{-1}$	1.52
7	1.09	$9.16 \cdot 10^{-1}$	$9.18 \cdot 10^{-1}$	$5.21 \cdot 10^{-1}$	$9.30 \cdot 10^{-1}$	1.37
9	$5.95 \cdot 10^{-1}$	$8.98 \cdot 10^{-1}$	$8.99 \cdot 10^{-1}$	$3.73 \cdot 10^{-1}$	$9.12 \cdot 10^{-1}$	1.23
10^2	$7.46 \cdot 10^{-4}$	$3.28 \cdot 10^{-1}$	$4.00 \cdot 10^{-1}$	$7.46 \cdot 10^{-4}$	$3.94 \cdot 10^{-1}$	$3.93 \cdot 10^{-1}$
10^3	$7.87 \cdot 10^{-7}$	$1.05 \cdot 10^{-6}$	$1.53 \cdot 10^{-2}$	$7.87 \cdot 10^{-7}$	$6.48 \cdot 10^{-2}$	$6.48 \cdot 10^{-2}$
10^4	$7.92 \cdot 10^{-10}$	$7.92 \cdot 10^{-10}$	$6.54 \cdot 10^{-3}$	$7.92 \cdot 10^{-10}$	$2.34 \cdot 10^{-9}$	$2.13 \cdot 10^{-9}$
10^5	$7.92 \cdot 10^{-13}$	$7.92 \cdot 10^{-13}$	$6.51 \cdot 10^{-3}$	$7.92 \cdot 10^{-13}$	$1.29 \cdot 10^{-12}$	$1.08 \cdot 10^{-12}$
10^7	$7.92 \cdot 10^{-19}$	–	–	$7.92 \cdot 10^{-19}$	$1.22 \cdot 10^{-18}$	$9.20 \cdot 10^{-19}$

Table 3: Pareto-like case: $q = 0.01$, $\alpha = 3$, $\beta = 0.5$

x	$T_{EV}(x)$	$T^-_{DG}(x)$	$T^+_{DG}(x)$	$T^-(x)$	$T^+_{C1}(x)$	$T^+_{C2}(x)$
1	$3.64 \cdot 10^{-2}$	$3.65 \cdot 10^{-2}$	$3.65 \cdot 10^{-2}$	$3.51 \cdot 10^{-2}$	$4.83 \cdot 10^{-2}$	$4.46 \cdot 10^{-2}$
3	$6.77 \cdot 10^{-3}$	$7.63 \cdot 10^{-3}$	$7.64 \cdot 10^{-3}$	$6.72 \cdot 10^{-3}$	$1.38 \cdot 10^{-2}$	$1.18 \cdot 10^{-2}$
5	$1.93 \cdot 10^{-3}$	$2.25 \cdot 10^{-3}$	$2.26 \cdot 10^{-3}$	$1.92 \cdot 10^{-3}$	$6.80 \cdot 10^{-3}$	$6.73 \cdot 10^{-3}$
7	$7.07 \cdot 10^{-4}$	$8.25 \cdot 10^{-4}$	$8.28 \cdot 10^{-4}$	$7.06 \cdot 10^{-4}$	$3.63 \cdot 10^{-3}$	$3.54 \cdot 10^{-3}$
9	$3.06 \cdot 10^{-4}$	$3.54 \cdot 10^{-4}$	$3.55 \cdot 10^{-4}$	$3.06 \cdot 10^{-4}$	$1.89 \cdot 10^{-3}$	$1.82 \cdot 10^{-3}$
10^2	$9.35 \cdot 10^{-9}$	$9.45 \cdot 10^{-9}$	$9.47 \cdot 10^{-9}$	$9.35 \cdot 10^{-9}$	$2.70 \cdot 10^{-8}$	$2.66 \cdot 10^{-8}$
10^3	$1.12 \cdot 10^{-13}$	$1.12 \cdot 10^{-13}$	$1.12 \cdot 10^{-13}$	$1.12 \cdot 10^{-13}$	$2.30 \cdot 10^{-13}$	$2.17 \cdot 10^{-13}$
10^4	$1.14 \cdot 10^{-18}$	–	–	$1.14 \cdot 10^{-18}$	$2.31 \cdot 10^{-18}$	$1.80 \cdot 10^{-18}$
10^5	$1.14 \cdot 10^{-23}$	–	–	$1.14 \cdot 10^{-23}$	$2.17 \cdot 10^{-23}$	$1.76 \cdot 10^{-23}$
10^7	$1.14 \cdot 10^{-33}$	–	–	$1.14 \cdot 10^{-33}$	$2.68 \cdot 10^{-33}$	$1.55 \cdot 10^{-33}$

Table 4: Pareto-like case: $q = 0.9$, $\alpha = 5$, $\beta = 0.25$

x	$T_{EV}(x)$	$T^-_{DG}(x)$	$T^+_{DG}(x)$	$T^-(x)$	$T^+_{C1}(x)$	$T^+_{C2}(x)$
1	2.95	$8.02 \cdot 10^{-1}$	$8.02 \cdot 10^{-1}$	$7.47 \cdot 10^{-1}$	$8.40 \cdot 10^{-1}$	1.06
3	$5.48 \cdot 10^{-1}$	$6.52 \cdot 10^{-1}$	$6.53 \cdot 10^{-1}$	$3.54 \cdot 10^{-1}$	$7.14 \cdot 10^{-1}$	$8.90 \cdot 10^{-1}$
5	$1.56 \cdot 10^{-1}$	$5.33 \cdot 10^{-1}$	$5.36 \cdot 10^{-1}$	$1.35 \cdot 10^{-1}$	$5.99 \cdot 10^{-1}$	$6.59 \cdot 10^{-1}$
7	$5.72 \cdot 10^{-2}$	$4.37 \cdot 10^{-1}$	$4.41 \cdot 10^{-1}$	$5.41 \cdot 10^{-2}$	$4.99 \cdot 10^{-1}$	$5.17 \cdot 10^{-1}$
9	$2.48 \cdot 10^{-2}$	$3.59 \cdot 10^{-1}$	$3.64 \cdot 10^{-1}$	$2.42 \cdot 10^{-2}$	$4.15 \cdot 10^{-1}$	$4.18 \cdot 10^{-1}$
10^2	$7.58 \cdot 10^{-7}$	$2.89 \cdot 10^{-5}$	$1.30 \cdot 10^{-4}$	$7.58 \cdot 10^{-7}$	$1.41 \cdot 10^{-4}$	$1.42 \cdot 10^{-4}$
10^3	$9.03 \cdot 10^{-12}$	$9.34 \cdot 10^{-12}$	$1.14 \cdot 10^{-11}$	$9.03 \cdot 10^{-12}$	$2.86 \cdot 10^{-11}$	$2.22 \cdot 10^{-11}$
10^4	$9.20 \cdot 10^{-17}$	–	–	$9.20 \cdot 10^{-17}$	$1.83 \cdot 10^{-16}$	$1.64 \cdot 10^{-16}$
10^5	$9.21 \cdot 10^{-22}$	–	–	$9.21 \cdot 10^{-22}$	$1.72 \cdot 10^{-21}$	$1.42 \cdot 10^{-21}$
10^7	$9.22 \cdot 10^{-32}$	–	–	$9.22 \cdot 10^{-32}$	$1.54 \cdot 10^{-31}$	$1.25 \cdot 10^{-31}$

Table 5: Pareto-like case: $q = 0.1$, $\alpha = 5$, $\beta = 0.25$

x	$T_{EV}(x)$	$T^-_{DG}(x)$	$T^+_{DG}(x)$	$T^-(x)$	$T^+_{C1}(x)$	$T^+_{C2}(x)$
1	32.4	$9.79 \cdot 10^{-1}$	$9.79 \cdot 10^{-1}$	$9.70 \cdot 10^{-1}$	$9.85 \cdot 10^{-1}$	1.36
3	6.03	$9.58 \cdot 10^{-1}$	$9.58 \cdot 10^{-1}$	$8.58 \cdot 10^{-1}$	$9.67 \cdot 10^{-1}$	1.61
5	1.72	$9.39 \cdot 10^{-1}$	$9.39 \cdot 10^{-1}$	$6.32 \cdot 10^{-1}$	$9.49 \cdot 10^{-1}$	1.49
7	$6.30 \cdot 10^{-1}$	$9.20 \cdot 10^{-1}$	$9.21 \cdot 10^{-1}$	$3.86 \cdot 10^{-1}$	$9.31 \cdot 10^{-1}$	1.27
9	$2.73 \cdot 10^{-1}$	$9.01 \cdot 10^{-1}$	$9.03 \cdot 10^{-1}$	$2.15 \cdot 10^{-1}$	$9.14 \cdot 10^{-1}$	1.10
10^2	$8.33 \cdot 10^{-6}$	$3.28 \cdot 10^{-1}$	$4.00 \cdot 10^{-1}$	$8.33 \cdot 10^{-6}$	$3.95 \cdot 10^{-1}$	$3.95 \cdot 10^{-1}$
10^3	$9.94 \cdot 10^{-11}$	–	–	$9.94 \cdot 10^{-11}$	$2.83 \cdot 10^{-4}$	$2.83 \cdot 10^{-4}$
10^4	$1.01 \cdot 10^{-15}$	–	–	$1.01 \cdot 10^{-15}$	$5.69 \cdot 10^{-15}$	$4.06 \cdot 10^{-15}$
10^5	$1.01 \cdot 10^{-20}$	–	–	$1.01 \cdot 10^{-20}$	$2.87 \cdot 10^{-20}$	$2.55 \cdot 10^{-20}$
10^7	$1.01 \cdot 10^{-30}$	–	–	$1.01 \cdot 10^{-30}$	$1.39 \cdot 10^{-30}$	$1.41 \cdot 10^{-30}$

Table 6: Pareto-like case: $q = 0.01$, $\alpha = 5$, $\beta = 0.25$

(2.64) and (2.65) respectively.

Tables 2 and 5 (where $q = 0.1$ that is, the relative safety loading is 0.11) indicate that the approximation T_{EV} is poor up to $x = 10^2$. And this effect is more evident from Tables 3 and 6 where T_{EV} lays far from real values of $T(x)$. All these results agree with the numerical calculations made by J. Cai and J. Garrido in [4] for the Pareto-like distribution (with $\alpha = 2$) and integrated lognormal tail distribution where they also came to the conclusion that the lower bound is pretty close to T_{SE} but their assertion that this "indicates a greater accuracy of lower bounds" is wrong because this fact indicates only that the proposed lower bound and T_{SE} are close to each other and they both can be pretty far from the real value $T(x)$ as in Table 6.

In all cases indicated in Tables 1 through 6, we also obtained bounds T^+_{C1} and T^+_{C2} using sample values of X_i, $i = 1, \ldots, N$ (the volume N of the sampling was 5000). Our routine was as follows. First, we built the sample (or, empirical) d.f.

$$\hat{F}(x) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\{X_i \leq x\}}$$

and then used it in order to find ε_1 and ε_2 from equations (2.60) and (2.62) thus obtaining sample bounds $\hat{T}^+_{C1}$ and $\hat{T}^+_{C2}$. The results are very optimistic: in all cases, the difference between $\hat{T}^+_{C1}$ (resp., $\hat{T}^+_{C2}$) and T^+_{C1} (resp., T^+_{C2}) does not exceeds 15%. This confirms the robustness of the procedures proposed.

Note that in all cases the proposed bounds T^+_{C1} and T^+_{C2} have reasonable accuracy and can easily be calculated. But they have evident limitations as their calculation for arbitrarily large x is impossible whereas our asymptotically correct bounds can be calculated for *any* x. Nevertheless, they can be used successfully for calculating $C^*[R(x^*, x^*]$ and, as the consequence, for obtaining asymptotically correct bounds. In accordance with the constructions of subsections 2.2 and 2.4 we took

$$R(x) = x^{1/(\alpha+1)}, \quad g(x) = x^{-\alpha/(\alpha+1)}.$$

Table 7 contains the following information:

x^* – the value defined in Lemma 2 which indicates the left abscissa of validity of our constructions;

C^+ – the constant defined in Theorem 3 which appears in the desired rate of convergence;

x^{**} – the value of x such that $C^+ g(x^{**}) = 1$;

$T^+(x^{**})$ – the actual value of the asymptotically correct upper bound at x^{**}.

α	q	x^*	C^+	x^{**}	$T^+(x^{**})$
	0.9	2.52	16.8	43.2	$1.55 \cdot 10^{-5}$
3	0.1	$1.18 \cdot 10^3$	$1.48 \cdot 10^7$	$3.63 \cdot 10^9$	$2.58 \cdot 10^{-27}$
	0.01	$1.24 \cdot 10^5$	$0.66 \cdot 10^{10}$	$1.35 \cdot 10^{13}$	$5.51 \cdot 10^{-37}$
	0.9	2.31	48.9	107.7	$2.69 \cdot 10^{-9}$
5	0.1	$1.44 \cdot 10^3$	$4.19 \cdot 10^6$	$9.21 \cdot 10^7$	$1.94 \cdot 10^{-36}$
	0.01	$1.65 \cdot 10^5$	-	-	-

Table 7: Pareto-like distribution.

It is necessary to clarify the meaning of x^{**}: we guarantee that the relative accuracy does not exceed 100% if $x \geq x^{**}$ and, conventionally, this value x^{**} can be viewed as the left abscissa of true applicability of the asymptotic approximation T_{EV}.

Comparing data from Table 7 with those from Tables 1 to 6, one can see that the value x^* clearly indicates the barrier where T_{EV} starts working. The value x^{**} is too pessimistic and we can explain its large values by the fact that the construction of $\beta_F(x)$ uses the shape of $F^c(x)$ for *large* x but not the values of $F^c(x)$ for *all* x. This is exactly the consequence of the tail-equivalence property indicated in subsection 2.2. The values of x^*, x^{**}, and C^+ can be diminished, if we take into account the values of $F^c(x)$ for comparatively small x (which seems to be difficult).

3.2 Weibull distribution

Tables 8 to 11 have the same structure as Tables 1 to 6 but they refer to the Weibull distribution (see Example 2 in subsection 2.3) where the parameters β and λ are chosen in such a way that the mean of F is 1. Its tail is lighter than that of the Pareto distribution since all its power moments exist.

Tables 8 and 9 contain figures for the case $\beta = 0.9$. This distribution has a relatively light tail as it is close in some sense ($\beta = 0.9 \approx 1$) to the exponential distribution. One can observe that the approximation T_{EV} is extremely poor.

The situation in Tables 10 and 11 illustrates another extreme. Here, $\beta = 0.1$ and the tail of F is rather heavy. One can notice that all approximations have good accuracy. In this case, the routine from [7] fails to work for comparatively large x unlike the proposed bounds T^+_{C1} and T^+_{C2} which are stable and fairly good in all cases.

This effect can be confirmed by the numerical examples from [4]. The authors of [4] obtained that, for the Pareto-like distribution (they chose $\alpha = 2$ and therefore, its tail is fairly heavy), the difference between upper and lower bounds is 15% - 30 %, whereas, for the integrated lognormal tail distribution (which tail is much lighter), that difference attains 2000%!

Let us highlight additionally the effect of the choice of $C^*[R(x^*), x^*]$. In the

x	$T_{EV}(x)$	$T_{DG}^-(x)$	$T_{DG}^+(x)$	$T^-(x)$	$T_{C1}^+(x)$	$T_{C2}^+(x)$
1	$3.51 \cdot 10^{-1}$	$2.92 \cdot 10^{-1}$	$2.92 \cdot 10^{-1}$	$2.60 \cdot 10^{-1}$	$3.57 \cdot 10^{-1}$	$3.59 \cdot 10^{-1}$
5	$1.16 \cdot 10^{-2}$	$4.41 \cdot 10^{-2}$	$4.46 \cdot 10^{-2}$	$1.15 \cdot 10^{-2}$	$7.90 \cdot 10^{-2}$	$7.39 \cdot 10^{-2}$
10	$2.44 \cdot 10^{-4}$	$4.34 \cdot 10^{-3}$	$4.52 \cdot 10^{-3}$	$2.44 \cdot 10^{-4}$	$9.56 \cdot 10^{-3}$	$9.00 \cdot 10^{-3}$
50	$4.19 \cdot 10^{-16}$	$2.98 \cdot 10^{-11}$	$7.59 \cdot 10^{-11}$	$4.19 \cdot 10^{-16}$	$1.13 \cdot 10^{-10}$	$1.13 \cdot 10^{-10}$
100	$2.01 \cdot 10^{-29}$	–	–	$2.01 \cdot 10^{-29}$	$1.34 \cdot 10^{-20}$	$1.34 \cdot 10^{-20}$
500	$7.05 \cdot 10^{-123}$	–	–	$7.05 \cdot 10^{-123}$	$1.69 \cdot 10^{-99}$	$1.69 \cdot 10^{-99}$
1000	$1.14 \cdot 10^{-228}$	–	–	$1.14 \cdot 10^{-228}$	$1.83 \cdot 10^{-199}$	$1.83 \cdot 10^{-199}$

Table 8: Weibull distribution: $q = 0.5$, $\beta = 0.9$, $\lambda = 1.04723$

x	$T_{EV}(x)$	$T_{DG}^-(x)$	$T_{DG}^+(x)$	$T^-(x)$	$T_{C1}^+(x)$	$T_{C2}^+(x)$
1	6.67	$8.99 \cdot 10^{-1}$	$8.99 \cdot 10^{-1}$	$8.70 \cdot 10^{-1}$	$9.19 \cdot 10^{-1}$	1.20
5	$2.20 \cdot 10^{-1}$	$7.35 \cdot 10^{-1}$	$7.37 \cdot 10^{-1}$	$1.80 \cdot 10^{-1}$	$7.77 \cdot 10^{-1}$	$9.04 \cdot 10^{-1}$
10	$4.64 \cdot 10^{-3}$	$5.71 \cdot 10^{-1}$	$5.77 \cdot 10^{-1}$	$4.61 \cdot 10^{-3}$	$6.10 \cdot 10^{-1}$	$6.13 \cdot 10^{-1}$
50	$7.96 \cdot 10^{-15}$	$6.97 \cdot 10^{-2}$	$8.82 \cdot 10^{-2}$	$7.96 \cdot 10^{-15}$	$1.02 \cdot 10^{-1}$	$1.02 \cdot 10^{-1}$
100	$3.82 \cdot 10^{-28}$	$4.02 \cdot 10^{-3}$	$1.02 \cdot 10^{-2}$	$3.82 \cdot 10^{-28}$	$7.18 \cdot 10^{-3}$	$7.18 \cdot 10^{-3}$
500	$1.34 \cdot 10^{-121}$	$5.37 \cdot 10^{-16}$	$1.24 \cdot 10^{-7}$	$1.34 \cdot 10^{-121}$	$1.17 \cdot 10^{-10}$	$1.17 \cdot 10^{-10}$
1000	$2.16 \cdot 10^{-227}$	–	–	$2.16 \cdot 10^{-227}$	$3.67 \cdot 10^{-21}$	$3.67 \cdot 10^{-21}$

Table 9: Weibull distribution: $q = 0.05$, $\beta = 0.9$, $\lambda = 1.04723$

x	$T_{EV}(x)$	$T_{DG}^-(x)$	$T_{DG}^+(x)$	$T^-(x)$	$T_{C1}^+(x)$	$T_{C2}^+(x)$
1	$1.08 \cdot 10^{-2}$	$1.07 \cdot 10^{-2}$	$1.08 \cdot 10^{-2}$	$1.07 \cdot 10^{-2}$	$1.24 \cdot 10^{-2}$	$1.16 \cdot 10^{-2}$
10^1	$3.34 \cdot 10^{-3}$	$3.34 \cdot 10^{-3}$	$3.35 \cdot 10^{-3}$	$3.33 \cdot 10^{-3}$	$3.85 \cdot 10^{-3}$	$3.65 \cdot 10^{-3}$
10^2	$7.64 \cdot 10^{-4}$	$7.64 \cdot 10^{-4}$	$7.66 \cdot 10^{-4}$	$7.63 \cdot 10^{-4}$	$9.14 \cdot 10^{-4}$	$8.37 \cdot 10^{-4}$
10^3	$1.19 \cdot 10^{-4}$	$1.19 \cdot 10^{-4}$	$1.20 \cdot 10^{-4}$	$1.19 \cdot 10^{-4}$	$1.41 \cdot 10^{-4}$	$1.31 \cdot 10^{-4}$
10^4	$1.15 \cdot 10^{-5}$	$1.15 \cdot 10^{-5}$	$1.15 \cdot 10^{-5}$	$1.15 \cdot 10^{-5}$	$1.32 \cdot 10^{-5}$	$1.27 \cdot 10^{-5}$
10^5	$6.03 \cdot 10^{-7}$	$6.03 \cdot 10^{-7}$	$6.07 \cdot 10^{-7}$	$6.03 \cdot 10^{-7}$	$7.21 \cdot 10^{-7}$	$6.67 \cdot 10^{-7}$
10^6	$1.48 \cdot 10^{-8}$	$1.48 \cdot 10^{-8}$	$1.49 \cdot 10^{-8}$	$1.48 \cdot 10^{-8}$	$1.85 \cdot 10^{-8}$	$1.63 \cdot 10^{-8}$
10^7	$1.39 \cdot 10^{-10}$	$1.39 \cdot 10^{-10}$	$1.40 \cdot 10^{-10}$	$1.39 \cdot 10^{-10}$	$1.60 \cdot 10^{-10}$	$1.55 \cdot 10^{-10}$
10^8	$3.89 \cdot 10^{-13}$	$3.89 \cdot 10^{-13}$	$3.94 \cdot 10^{-13}$	$3.89 \cdot 10^{-13}$	$4.65 \cdot 10^{-13}$	$4.35 \cdot 10^{-13}$
10^9	$2.38 \cdot 10^{-16}$	–	–	$2.38 \cdot 10^{-16}$	$2.98 \cdot 10^{-16}$	$2.70 \cdot 10^{-16}$
10^{10}	$2.15 \cdot 10^{-20}$	–	–	$2.15 \cdot 10^{-20}$	$2.85 \cdot 10^{-20}$	$2.44 \cdot 10^{-20}$

Table 10: Weibull distribution: $q = 0.5$, $\beta = 0.1$, $\lambda = 4.52874$

x	$T_{EV}(x)$	$T^-_{DG}(x)$	$T^+_{DG}(x)$	$T^-(x)$	$T^+_{C1}(x)$	$T^+_{C2}(x)$
1	$2.05 \cdot 10^{-1}$	$1.82 \cdot 10^{-1}$	$1.87 \cdot 10^{-1}$	$1.70 \cdot 10^{-1}$	$2.26 \cdot 10^{-1}$	$2.08 \cdot 10^{-1}$
10^1	$6.35 \cdot 10^{-2}$	$6.31 \cdot 10^{-2}$	$6.58 \cdot 10^{-2}$	$5.97 \cdot 10^{-2}$	$8.40 \cdot 10^{-2}$	$7.31 \cdot 10^{-2}$
10^2	$1.45 \cdot 10^{-2}$	$1.47 \cdot 10^{-2}$	$1.56 \cdot 10^{-2}$	$1.43 \cdot 10^{-2}$	$1.93 \cdot 10^{-2}$	$1.71 \cdot 10^{-2}$
10^3	$2.26 \cdot 10^{-3}$	$2.28 \cdot 10^{-3}$	$2.46 \cdot 10^{-3}$	$2.26 \cdot 10^{-3}$	$2.98 \cdot 10^{-3}$	$2.65 \cdot 10^{-3}$
10^4	$2.18 \cdot 10^{-4}$	$2.18 \cdot 10^{-4}$	$2.40 \cdot 10^{-4}$	$2.18 \cdot 10^{-4}$	$2.81 \cdot 10^{-4}$	$2.51 \cdot 10^{-4}$
10^5	$1.15 \cdot 10^{-5}$	$1.15 \cdot 10^{-5}$	$1.29 \cdot 10^{-5}$	$1.15 \cdot 10^{-5}$	$1.38 \cdot 10^{-5}$	$1.31 \cdot 10^{-5}$
10^6	$2.81 \cdot 10^{-7}$	$2.81 \cdot 10^{-7}$	$3.27 \cdot 10^{-7}$	$2.81 \cdot 10^{-7}$	$3.47 \cdot 10^{-7}$	$3.17 \cdot 10^{-7}$
10^7	$2.64 \cdot 10^{-9}$	$2.64 \cdot 10^{-9}$	$3.21 \cdot 10^{-9}$	$2.64 \cdot 10^{-9}$	$3.15 \cdot 10^{-9}$	$3.01 \cdot 10^{-9}$
10^8	$7.40 \cdot 10^{-12}$	$7.40 \cdot 10^{-12}$	$1.64 \cdot 10^{-11}$	$7.40 \cdot 10^{-12}$	$9.69 \cdot 10^{-12}$	$8.35 \cdot 10^{-12}$
10^9	$4.53 \cdot 10^{-15}$	$4.06 \cdot 10^{-15}$	$6.92 \cdot 10^{-12}$	$4.53 \cdot 10^{-15}$	$5.55 \cdot 10^{-15}$	$5.11 \cdot 10^{-15}$
10^{10}	$4.08 \cdot 10^{-19}$	–	–	$4.08 \cdot 10^{-19}$	$6.41 \cdot 10^{-19}$	$4.58 \cdot 10^{-19}$

Table 11: Weibull distribution: $q = 0.05$, $\beta = 0.1$, $\lambda = 4.52874$

examples above we used bounds (2.64) and (2.65) for this purpose. Now we illustrate the difference caused by a good or bad estimate of $C^*[R(x^*), x^*]$. We have already mentined that it is possible to give crude analytical bounds for $C^*[R(x^*), x^*]$. One of such bounds was proposed in [27]:

$$C^*[R(x^*), x^*] \le C_1^* \equiv \frac{\delta^+(1-q)^2}{F^c(x^*)g(x^*)},$$

where δ^+ and g are taken from Lemma 2. Another bound for $C^*[R(x^*), x^*]$ can be found if one applies upper estimates $T^+_{GS}(x)$ of $T(x)$ over $[R(x^*), x^*]$ taken from [15, Theorem 4.3.2]. Let us denote this bound by C_2^*:

$$C^*[R(x^*), x^*] \le C_2^*$$

(it is not necessary to write the bound explicitly here). And, at last, let us denote by C_3^* the bound obtained if one uses the estimate $T^+_{C1}(x)$ for $x \in [R(x^*), x^*]$ which has already been done in the preceding examples.

In accordance with this, let C_i^+, $i = 1, 2, 3$, be the corresponding constants calculated by formula (2.48) where $C^*[R(x^*), x^*]$ is replaced by its estimates C_i^*, $i = 1, 2, 3$, and denote by $T_i^+(x)$ the corresponding asymptotically correct upper bounds having the form

$$T_i^+(x) = T_{EV}(x)(1 + C_i^+ g(x)).$$

For illustrations, we took $q = 0.5$ which is typical for actuarial applications and we chose numerically the optimal value δ^+ which turned out to be 0.999. The corresponding constants C_i^+ have the following values:

$$C_1^+ = 4.9 \cdot 10^5, \quad C_2^+ = 3.9 \cdot 10^3, \quad C_1^+ = 3.6 \cdot 10^3.$$

x	$T_{EV}(x)$	$T_1^+(x)$	$T_2^+(x)$	$T_3^+(x)$
$1.6 \cdot 10^2$	$1.3 \cdot 10^{-5}$	$2.1 \cdot 10^{-1}$	$1.6 \cdot 10^{-3}$	$1.5 \cdot 10^{-3}$
$3.5 \cdot 10^2$	$2.6 \cdot 10^{-6}$	$2.6 \cdot 10^{-2}$	$2.0 \cdot 10^{-4}$	$1.9 \cdot 10^{-4}$
$1.0 \cdot 10^3$	$3.1 \cdot 10^{-7}$	$1.5 \cdot 10^{-3}$	$1.2 \cdot 10^{-5}$	$1.2 \cdot 10^{-5}$
$2.5 \cdot 10^3$	$5.0 \cdot 10^{-8}$	$1.3 \cdot 10^{-4}$	$1.1 \cdot 10^{-6}$	$1.0 \cdot 10^{-6}$
$5.0 \cdot 10^3$	$1.2 \cdot 10^{-8}$	$2.1 \cdot 10^{-5}$	$1.7 \cdot 10^{-7}$	$1.6 \cdot 10^{-7}$

Table 12: Comparison of T_i^+, $i = 1, 2, 3$.

Let us call the reader's attention to the fact that C_1^+ is much larger than C_2^+ and C_3^+ which are close to each other. The numerical results are presented in Table 12.

One can see that the choice C_1^* yields poor bounds whereas both C_2^* and C_3^* give better results. This means that certain attention should be paid to this choice which affect the results. Let us mention also that, in order to find C_2^*, we do not have to solve any functional equation (like (2.60) or (2.62)) and because of this, it can have a computational advantage in practical situations.

4 Conclusion

During this research we

(i) provided quantitative accuracy estimates of the known asymptotic approximations of the probability of ruin and the distribution of accumulated claims in the presence of large claims;

(ii) proposed operational bounds of the mentioned characteristics.

Most of the results were obtained by mathematical methods which were specially developed for the purposes of the research. All the methods proposed can be implemented into software packages. We considered also various example illustrating both the technique and the quality of the estimates. To be more specific, the following results should be considered as basic.

1. New characterizations of SE-distributions are proposed (see Section 2). These characterizations can be used for solving various applied problems (from risk theory, queueing theory, etc.).

2. We derived asymptotically correct two-sided bounds (which have been unknown in the literature until now) for geometric (Section 3) and general (Section 4) random sums with subexponential summands. These results are mathematically interesting and give insight into the behavior of random sums with heavy-tailed

summands. These sums can be interpreted as accumulated claim sizes of an insurance company or the probability of its ruin.

3. The proposed bounds give us the possibility to detect the critical value x^*, after which the known asymptotic approximations can be applied with a reasonable accuracy (see Section 3). These bounds seem to be rather pessimistic for actuarial practice (although, their application to communication networks and some queueing problems where the loss probability should be less than the probability of ruin is possible). This fact can be explained by the tail-equivalence property of SE-distributions (see Subsection 1.2): SE-effects can only occur for large x. Numerical examples collected in Section 3 show that the asymptotic approximations can differ significantly from real values. So, the accuracy properties (indicated above) of the bounds has solid foundation and is implied by the properties of SE-distributions.

   The fact is that the asymptotic formulas can approximate the real distributions poorly and therefore, their usage in actuarial practice is questionable. Our research reveals the situations where these formulas can be applied and where they cannot.

4. We propose, for practical needs, two routines giving close results which provide us with upper bounds of the ruin probability (see Subsection 2.7). These upper bounds turned out to be rather accurate (as they are derived from the exact representation of the probability of ruin) and can be used in practice. The corresponding numerical methods found in standard packages allow us to solve functional equations and make integration. Furthermore, these routines work in the case where the exact form of F is unknown and only samples of X_i are given. The accuracy of calculation is demonstrated in Section 3.

5. This research provided us with a clear qualitative picture of the limiting behavior of random sums with SE-summands. The existing asymptotic approximations cannot be recommended as approximations of real distributions if q is small (or, in general, if ν is large) or if the tail $F^c(x)$ is not sufficiently heavy. In these cases, it is better to use $T^+_{C1}(x)$ or $T^+_{C2}(x)$, or the approximations taken from [15]. In other cases, the asymptotic approximations can work accurately. Our practical recommendations are as follows. Calculate $T^-(x)$ and any of $T^+_{C1}(x)$ or $T^+_{C2}(x)$. If these bounds are close to each other, it is possible to use the asymptotic approximations. If they are far from each other, then the approximations can give enormous errors and it is better to use upper bounds $T^+_{C1}(x)$ or $T^+_{C2}(x)$.

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